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Albert Einstein, Leopold Infeld and Banesh Hoffmann (1938): The Gravitational Equations and the Problem of Motion
DOI: 10.34663/9783945561317-17

In: Alexander S. Blum and Dean Rickles (eds.): *Quantum Gravity in the First Half of the Twentieth Century : A Sourcebook*
Online version at https://edition-open-sources.org/sources/10/

ISBN 978-3-945561-31-7, DOI 10.34663/9783945561317-00
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The Deutsche Nationalbibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data are available in the Internet at http://dnb.d-nb.de

Chapter 15
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Albert Einstein, Leopold Infeld and Banesh Hoffmann (1938). The Gravitational Equations and the Problem of Motion. *The Annals of Mathematics*, 39: 65–100.

ANNALS OF MATHEMATICS
Vol. 39, No. 1, January, 1938

THE GRAVITATIONAL EQUATIONS AND THE PROBLEM OF MOTION

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(Received June 16, 1937)

Introduction. In this paper we investigate the fundamentally simple question of the extent to which the relativistic equations of gravitation determine the motion of ponderable bodies.

Previous attacks on this problem[1] have been based upon gravitational equations in which some specific energy-momentum tensor for matter has been assumed. Such energy-momentum tensors, however, must be regarded as purely temporary and more or less phenomenological devices for representing the structure of matter, and their entry into the equations makes it impossible to determine how far the results obtained are independent of the particular assumption made concerning the constitution of matter.

Actually, the only equations of gravitation which follow without ambiguity from the fundamental assumptions of the general theory of relativity are the equations for empty space, and it is important to know whether they *alone* are capable of determining the motion of bodies. The answer to this question is not at all obvious. It is possible to find examples in classical physics leading to either answer, yes or no. For instance, in the ordinary Maxwell equations for empty space, in which electrical particles are regarded as point singularities of the field, the motion of these singularities is not determined by the linear field equations. On the other hand, the well-known theory of Helmholtz on the motion of vortices in a non-viscous fluid gives an instance where the motion of line singularities is actually determined by partial differential equations alone, which are there non-linear.

We shall show in this paper that the gravitational equations for empty space are in fact sufficient to determine the motion of matter represented as point singularities of the field. The gravitational equations are non-linear, and, because of the necessary freedom of choice of the coördinate system, are such that four differential relations exist between them so that they form an overdetermined system of equations. The overdetermination is responsible for the existence of equations of motion, and the non linear character for the existence of terms expressing the interaction of moving bodies.

Two essential steps lead to the determination of the motion.

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[1] Droste, Ac. van Wet. Amsterdam *19*, 447 (1916). De Sitter, Monthly Notices of the R. A. S. *67*, 155 (1916). Mathisson, Zeits. f. Physik, *67*, 270, 826 (1931), *69*, 389 (1931). Levi-Civita, Am. Jour. of Math., lix, 3, 225 (1937).

(1) By means of a new method of approximation, specially suited to the treatment of quasi-stationary fields, the gravitational field due to moving particles is determined.
(2) It is shown that for two-dimensional spatial surfaces containing singularities certain surface integral conditions are valid which determine the motion.

In the second part of this paper we actually calculate the first two non-trivial stages of the approximation. In the first of these the equations of motion take the Newtonian form. In the second the equations of motion, which we calculate only for the case of two massive particles, take a more complicated form but do not involve third or higher derivatives with respect to the time.

The method is, in principle, applicable for any order of approximation, the problem reducing to specific integrations at each stage, but we have not proved that higher time derivatives than the second will not ultimately occur in the equations of motion.

In the determination of the field and the equations of motion non-Galilean values at infinity and singularities of the type of dipoles, quadrupoles, and higher poles, must be excluded from the field in order that the solution shall be unique.

It is of significance that our equations of motion do not restrict the motion of the singularities more strongly than the Newtonian equations, but this may be due to our simplifying assumption that matter is represented by singularities, and it is possible that it would not be the case if we could represent matter in terms of a field theory from which singularities were excluded. The representation of matter by means of singularities does not enable the field equations to fix the sign of mass so that, so far as the present theory is concerned, it is only by convention that the interaction between two bodies is always an attraction and not a repulsion. A possible clue as to why the mass must be positive can be expected only from a theory which gives a representation of matter free from singularities.[2]

Our method can be applied to the case when the Maxwell energy-momentum tensor is included in the field equations and, as is shown in part II, it leads to a derivation of the Lorentz force.

In the Maxwell-Lorentz electrodynamics, as also in the earlier approximation method for the solution of the gravitational equations, the problem of determining the field due to moving bodies is solved through the integration of wave equations by retarded potentials. The sign of the flow of time there plays a decisive rôle since, in a certain sense, the field is expanded in terms of only those waves which proceed towards infinity. In our theory, however, the equations to be solved at each stage of the approximation are not wave equations but merely spatial potential equations. Since such equations as those of the gravitational and of the electromagnetic field are actually invariant under a

[2] Einstein and Rosen, Phys. Rev. vol. 48, 73 (1935).

reversal of the sign of time, it would seem that the method presented here, is the natural one for their solution. Our method, in which the time direction is not distinguished, corresponds to the introduction of standing waves in the wave equation and cannot lead to the conclusion that in the circular motion of two point masses energy is radiated to infinity in the form of waves.

I. General Theory

1. **Field Equations and Coördinate Conditions.** Since it is an essential part of the work to make a separation between space and time we shall, throughout this paper, use the convention that Latin indices take on only the spatial values 1, 2, 3 while Greek indices refer to both space and time, running over the values 0, 1, 2, 3.

As explained in the introduction, we discuss only the gravitational equations for empty space, treating the sources of the field as singularities. If we denote the ordinary derivative of a quantity by means of a line followed by the appropriate suffix, as

$$(1,1) \qquad \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \rightarrow g_{\mu\nu|\sigma}\,; \qquad \frac{\partial^2 g_{\mu\nu}}{\partial x^\sigma \partial x^\rho} \rightarrow g_{\mu\nu|\sigma\rho}\,,$$

we may write the field equations in the form

$$(1,2) \qquad R_{\mu\nu} = -\left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}_{|\lambda} + \left\{\begin{matrix}\lambda\\ \mu\lambda\end{matrix}\right\}_{|\nu} + \left\{\begin{matrix}\lambda\\ \mu\sigma\end{matrix}\right\}\left\{\begin{matrix}\sigma\\ \lambda\nu\end{matrix}\right\} - \left\{\begin{matrix}\lambda\\ \mu\nu\end{matrix}\right\}\left\{\begin{matrix}\sigma\\ \lambda\sigma\end{matrix}\right\} = 0.$$

Let the symbols $\eta_{\mu\nu}$, $\eta^{\mu\nu}$ be defined by

$$(1,3) \qquad \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{Bmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Bmatrix},$$

so that they represent the metric of empty space-time. Then if we introduce the quantities $h_{\mu\nu}$, $h^{\mu\nu}$ by the relations

$$(1,4) \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu},$$

the $h_{\mu\nu}$ and $h^{\mu\nu}$ will represent the deviation of space-time from the flat case. The $h^{\mu\nu}$ can be calculated as functions of the $h_{\mu\nu}$ by means of the relations

$$(1,5) \qquad g_{\mu\nu} g^{\nu\sigma} = \delta^\sigma_\mu\,.$$

In general the $h_{\mu\nu}$ will be small relative to unity, but we make no assumptions here concerning their order of magnitude.

By means of (1, 4), (1, 5) we can express the components of $R_{\mu\nu}$ as functions of the $h_{\mu\nu}$, and for reasons which will become clear when we come to the method of approximation used in the present work we separate the various terms so obtained into two groups in the following manner. First we separate the terms linear in the h's from those which are quadratic and of higher order. At this stage of the separation the field equations are of the form

$$(1, 6) \quad R_{00} = \tfrac{1}{2}\{-h_{00|ss} + 2h_{0s|0s} - h_{ss|00}\} + L'_{00} = 0,$$

$$(1, 7) \quad R_{0n} = \tfrac{1}{2}\{-h_{0n|ss} + h_{0s|ns} + h_{ns|0s} - h_{ss|n0}\} + L'_{0n} = 0,$$

$$(1, 8) \quad R_{mn} = \tfrac{1}{2}\{-h_{mn|ss} + h_{ms|ns} + h_{ns|ms} - h_{ss|mn} + h_{mn|00} - h_{m0|n0} - h_{n0|m0} + h_{00|mn}\} + L'_{mn} = 0,$$

where the $L'_{\mu\nu}$ represent the non-linear terms. We now take

(1, 9) from R_{00} the terms $h_{0s|0s} - \frac{1}{2}h_{ss|00}$,

(1, 10) from R_{0n} no terms

(1, 11) and from R_{mn} the terms $-\frac{1}{2}h_{0m|0n} - \frac{1}{2}h_{0n|0m} + \frac{1}{2}h_{mn|00}$,

and add them to the non-linear group. Introducing the symbol $L_{\mu\nu}$ to denote the non-linear group $L'_{\mu\nu}$ together with these added linear members, we may write the field equations in the separated from

$$(1, 12) \quad R_{00} = -\tfrac{1}{2}h_{00|ss} + L_{00} = 0,$$

$$(1, 13) \quad \begin{aligned} R_{0n} = &-\tfrac{1}{2}h_{0n|ss} + \tfrac{1}{2}(h_{ns} - \tfrac{1}{2}\delta_{ns}h_{ll} + \tfrac{1}{2}\delta_{ns}h_{00})_{|0s} \\ &- \tfrac{1}{4}(h_{00} + h_{ss})_{|0n} + \tfrac{1}{2}h_{0s|ns} + L_{0n} = 0, \end{aligned}$$

$$(1, 14) \quad \begin{aligned} R_{mn} = &-\tfrac{1}{2}h_{mn|ss} + \tfrac{1}{2}(h_{ms} - \tfrac{1}{2}\delta_{ms}h_{ll} + \tfrac{1}{2}\delta_{ms}h_{00})_{|ns} \\ &+ \tfrac{1}{2}(h_{ns} - \tfrac{1}{2}\delta_{ns}h_{ll} + \tfrac{1}{2}\delta_{ns}h_{00})_{|ms} + L_{mn} = 0, \end{aligned}$$

where the L's are given explicitly by the formulas

$$(1, 15) \quad \begin{aligned} L_{00} = &\, h_{0s|0s} - \tfrac{1}{2}h_{ss|00} - (h^{\lambda\sigma}[00, \sigma])_{|\lambda} + (h^{\lambda\sigma}[0\lambda, \sigma])_{|0} \\ &+ \left\{ {\lambda \atop 0\sigma} \right\}\left\{ {\sigma \atop \lambda 0} \right\} - \left\{ {\lambda \atop 00} \right\}\left\{ {\sigma \atop \lambda\sigma} \right\}, \end{aligned}$$

$$(1, 16) \quad L_{0n} = -(h^{\lambda\sigma}[0n, \sigma])_{|\lambda} + (h^{\lambda\sigma}[n\lambda, \sigma])_{|0} + \left\{ {\lambda \atop n\sigma} \right\}\left\{ {\sigma \atop \lambda 0} \right\} - \left\{ {\lambda \atop n0} \right\}\left\{ {\sigma \atop \lambda\sigma} \right\},$$

$$(1, 17) \quad \begin{aligned} L_{mn} = &-\tfrac{1}{2}h_{0m|0n} - \tfrac{1}{2}h_{0n|0m} + \tfrac{1}{2}h_{mn|00} - (h^{\lambda\sigma}[mn, \sigma])_{|\lambda} \\ &+ (h^{\lambda\sigma}[m\lambda, \sigma])_{|n} + \left\{ {\lambda \atop m\sigma} \right\}\left\{ {\sigma \atop \lambda n} \right\} - \left\{ {\lambda \atop mn} \right\}\left\{ {\sigma \atop \lambda\sigma} \right\}. \end{aligned}$$

If we introduce the quantities $\gamma_{\mu\nu}$ defined by

$$(1, 18) \quad \gamma_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2}\,\eta_{\mu\nu}\,\eta^{\sigma\rho}\,h_{\sigma\rho}\,,$$

or, in expanded form,

$$\gamma_{00} = \tfrac{1}{2}h_{00} + \tfrac{1}{2}h_{ll}\,, \tag{1, 19}$$

$$\gamma_{0n} = h_{0n}\,, \tag{1, 20}$$

$$\gamma_{mn} = h_{mn} - \tfrac{1}{2}\delta_{mn}h_{ll} + \tfrac{1}{2}\delta_{mn}h_{00}\,, \tag{1, 21}$$

we may write the field equations (1, 12), (1, 13), (1, 14) in the form

$$R_{00} = -\tfrac{1}{2}h_{00|ss} + L_{00} = 0, \tag{1, 22}$$

$$R_{0n} = -\tfrac{1}{2}h_{0n|ss} + \tfrac{1}{2}\gamma_{ns|0s} + \tfrac{1}{2}(\gamma_{0s|s} - \gamma_{00|0})_{|n} + L_{0n} = 0, \tag{1, 23}$$

$$R_{mn} = -\tfrac{1}{2}h_{mn|ss} + \tfrac{1}{2}\gamma_{ms|sn} + \tfrac{1}{2}\gamma_{ns|sm} + L_{mn} = 0. \tag{1, 24}$$

Since there are four identities between these field equations, we may impose four coördinate conditions, in the form of four non-tensorial equations involving the gravitational potentials, so as to limit the arbitrariness of the solutions by limiting the freedom of choice of the coördinate system. It turns out to be simplest to use coördinate conditions which involve only quantities which enter the explicitly written parts of the field equations (1, 23), (1, 24). These equations, in fact, suggest that we take as our coördinate conditions[3]

$$\gamma_{0s|s} - \gamma_{00|0} = 0, \tag{1, 25}$$

$$\gamma_{ms|s} = 0. \tag{1, 26}$$

With these coördinate conditions the field equations become merely

$$h_{00|ss} = 2L_{00}\,, \tag{1, 27}$$

$$h_{0n|ss} = 2L_{0n}\,, \tag{1, 28}$$

$$h_{mn|ss} = 2L_{mn}\,. \tag{1, 29}$$

For the further argument it is necessary that we write these equations in such a way that the Laplacians of the γ's enter instead of the Laplacians of the h's. We therefore replace the above equations by the equivalent equations

$$\gamma_{00,ss} = 2\Lambda_{00}\,, \tag{1, 30}$$

$$\gamma_{0n,ss} = 2\Lambda_{0n}\,, \tag{1, 31}$$

$$\gamma_{mn,ss} = 2\Lambda_{mn}\,, \tag{1, 32}$$

---

[3] The choice of the coördinate conditions is, to a large extent arbitrary, and it might seem rather more natural to use the conditions

$$\eta^{\nu\mu}\gamma_{\alpha\nu|\mu} = 0$$

which are invariant under a Lorentz transformation. However, it turns out that the actual calculation of the field is simpler when we use the coördinate conditions given in the text and it is for this reason that we employ it in the general theory.

where Λ is related to L exactly as γ is to h:

$$\Lambda_{00} = \tfrac{1}{2}L_{00} + \tfrac{1}{2}L_{ll}\,, \tag{1, 33}$$

$$\Lambda_{0n} = L_{0n}\,, \tag{1, 34}$$

$$\Lambda_{mn} = L_{mn} - \tfrac{1}{2}\delta_{mn}L_{ll} + \tfrac{1}{2}\delta_{mn}L_{00}\,. \tag{1, 35}$$

These field equations, (1, 33), (1, 34), (1, 35) together with the coördinate conditions (1, 25), (1, 26) will form the basis of our further considerations.

2. **Fundamental Integral Properties of the Field.** Let us consider three functions A_n; ($n = 1, 2, 3$). They need not be tensors. From these functions we may build the three further functions

$$(A_{n|s} - A_{s|n})_{|s}\,, \tag{2, 1}$$

which can be explicitly written as

$$\begin{gathered}\{(A_{1|2} - A_{2|1})_{|2} - (A_{3|1} - A_{1|3})_{|3}\}, \quad \{(A_{2|3} - A_{3|2})_{|3} - (A_{1|2} - A_{2|1})_{|1}\}, \\ \{(A_{3|1} - A_{1|3})_{|1} - (A_{2|3} - A_{3|2})_{|2}\}.\end{gathered} \tag{2, 2}$$

These three functions thus constitute the curl of the three functions

$$(A_{2|3} - A_{3|2}), \qquad (A_{3|1} - A_{1|3}), \qquad (A_{1|2} - A_{2|1}). \tag{2, 3}$$

Consider any surface S which does not pass through singularities of the field. Since (2, 1) is the curl of (2, 3), it follows from Stokes' theorem that the integral of the "normal"[4] component of (2, 1) over S is equal to the line integral of the tangential component of (2, 3) taken around the rim of S. If S is a closed surface its rim is of zero length so that the latter integral will vanish. We therefore have the theorem that, if S is any closed surface which does not pass through singularities of the field, then

$$\int (A_{n|s} - A_{s|n})_{|s} \cos(\mathbf{n}\cdot\mathbf{N})\, dS = 0, \tag{2, 4}$$

where $(\mathbf{n}\cdot\mathbf{N})$ denotes the "angle" between the direction of x^n and the "normal" to S, and the summation convention applies to the n. This theorem is valid whether S encloses singularities or not, and we shall now apply it to the present problem.

[4] Words like *normal, angle, sphere,* and so on are used here in a purely conventional sense to designate the corresponding functions of the coördinates x^m and equations which are implied by these terms in Euclidean geometry. The argument of this paragraph is independent of any particular metric, and we use the Euclidean nomenclature merely because it is apt and convenient.

From the coördinate conditions (1, 25), (1, 26) and the field equations (1, 31), (1, 32) we have

(2, 5) $$(\gamma_{0n|s} - \gamma_{0s|n})_{|s} = 2\Lambda_{0n} - \gamma_{00|0n} ,$$

(2, 6) $$(\gamma_{mn|s} - \gamma_{ms|n})_{|s} = 2\Lambda_{mn} .$$

We see that the left-hand sides of (2, 5), (2, 6) give four quantities of the form (2, 1), one coming from (2, 5) and three from (2, 6) for $m = 1, 2, 3$. It follows from (2, 4) that, if S is a surface which does not pass through singularities of the field,

(2, 7) $$\int (\gamma_{00|0n} - 2\Lambda_{0n}) \cos (\mathbf{n} \cdot \mathbf{N})\, dS = 0,$$

(2, 8) $$\int 2\Lambda_{mn} \cos (\mathbf{n} \cdot \mathbf{N})\, dS = 0.$$

From (2, 5), (2, 6) we see that, in those regions where there are no singularities,

(2, 9) $$(\gamma_{00|0n} - 2\Lambda_{0n})_{|n} = 0,$$

(2, 10) $$(2\Lambda_{mn})_{|n} = 0.$$

Therefore Gauss' theorem shows that if we take two closed surfaces S, S' such that no singularity lies on or between S and S', the integrals over S and S' give the same result. But the validity of the integral conditions for surfaces which enclose singularities, or more generally, which enclose regions where the field equations for empty space are not fulfilled, can only be shown by means of Stokes' theorem.

We are treating matter as a singularity in the field. Let us assume there are p bodies, each represented by a point singularity. The coördinates of each such singularity will be functions of the time alone. Since (2, 7), (2, 8) are valid for any S provided only that it does not pass through a singularity, we may choose p such surfaces, each enclosing only one of the p singularities, and thus obtain $4p$ distinct integral conditions. Each of these, being now independent of the shape of its S, will give a relation between the coördinates of the singularities and their time derivatives, and we shall see later that the integral conditions give, in fact, the *equations of motion* of the singularities. These equations are derived here from the field equations and coördinate conditions alone without any extraneous assumption.

If, instead of integrating around one singularity at a time, we integrate over a surface which contains all the singularities, we obtain the laws of conservation of energy and linear momentum for the whole system. These laws are, of course, merely consequences of the laws of motion for the individual particles but owing to many cancellations they take a comparatively simple form.

3. **The Method of Approximation.** The method of approximation which has been used up to now in the theory of relativity is as follows. We consider that in the equation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3, 1}$$

the $h_{\mu\nu}$ depend continuously on a positive parameter λ in such a way that they vanish for $\lambda = 0$, so that for $\lambda = 0$ space-time becomes Galilean. We assume, therefore, that the $h_{\mu\nu}$ can be expanded in a power series[5] in λ:

$$h_{\mu\nu} = \sum_{l=1}^{\infty} \lambda^l \underset{l}{h_{\mu\nu}}. \tag{3, 2}$$

This expansion is introduced in the field equations which are then grouped according to the different powers of λ, taking the form

$$0 = R_{\mu\nu} = \sum_{l=1}^{\infty} \lambda^l \underset{l}{R_{\mu\nu}}. \tag{3, 3}$$

In order that a set of $h_{\mu\nu}$ depending on the parameter λ shall exist as a solution of the field equations it is necessary that each of the equations

$$\underset{l}{R_{\mu\nu}} = 0 \tag{3, 4}$$

shall be satisfied. The best known example of this method is its application to the first approximation.

We shall now show why this method of approximation is unsuitable for the treatment of quasi-stationary fields. If we introduce an energy tensor for the matter which produces the field we obtain for the first approximation, using imaginary time, the well-known equations

$$\gamma_{\mu\nu|\sigma\sigma} = -2T_{\mu\nu}, \tag{3, 5}$$

where the coördinate system is determined by the equations

$$\gamma_{\mu\sigma|\sigma} = 0. \tag{3, 6}$$

In the simplest case of incoherent matter (dust) producing the field we have

$$T_{\mu\nu} = \rho \frac{d\xi^\mu}{ds}\frac{d\xi^\nu}{ds}, \tag{3, 7}$$

where $d\xi^\mu/ds$ are the components of the velocity measured in terms of the proper time s. If we are concerned with a quasi-static situation, $d\xi^0/ds$ is of the order of magnitude of unity while the $d\xi^m/ds$ are relatively small. Thus in such a case we shall have

$$| T_{00} | \gg | T_{0n} | \gg | T_{mn} |, \tag{3, 8}$$

[5] In λ^l the l will always be an exponent, not a contravariant index!

and from the equations (3, 5) we must have correspondingly

$$|\gamma_{00}| \gg |\gamma_{0n}| \gg |\gamma_{mn}| . \tag{3, 9}$$

The usual method of approximation does not take this into account since it treats all the γ's as of the same order of magnitude although, in the quasi-static case, γ_{00} is very much larger than the other components of $\gamma_{\mu\nu}$. A really good method of approximation for the quasi-static case should make essential use of the relations (3, 9).

We are led to our present method of approximation most simply by considering the problem of constructing a method of approximation which is suitable for the solution of the approximate field equations (3, 5) for the quasi-static case. It turns out that the method of approximation to which we are led in this way is also suitable for the solution of the rigorous gravitational equations even when we are not dealing with quasi-static cases.

The first step is to give an explicit expression for the fact that the time derivative of a field quantity is small relative to the quantity itself and to its spatial derivatives. To do this we introduce an auxiliary time coördinate

$$\tau = \lambda x^0 \tag{3, 10}$$

and assume that every field quantity is a function of (τ, x^1, x^2, x^3) rather than of (x^0, x^1, x^2, x^3). If φ is such a quantity we now assume that φ, $\varphi_{|m}$ and $\partial\varphi/\partial\tau$ are of the same order of magnitude, so that $\varphi_{|0}$ is of the order of $\lambda\varphi$.

From this we conclude that if T_{00} in (3, 7) is of the order of magnitude of λ^q, then T_{0n} will be of the order of λ^{q+1} and T_{mn} of the order of λ^{q+2}.

Further, it follows from well-known considerations concerning the first approximation (the conservation of energy for the motion of a point) that γ_{00}, which is the potential energy of a unit mass, is of the same order of magnitude as the square of the velocity and is thus, in our present notation, of the order of λ^2. Hence we have the following orders of magnitude for the γ's:

$$\gamma_{00} \sim \lambda^2; \qquad \gamma_{0n} \sim \lambda^3; \qquad \gamma_{mn} \sim \lambda^4. \tag{3, 11}$$

If we expand the γ's as power series in λ we must therefore take the lowest powers of the expansions to be of the orders given in (3, 11). The fact that only second derivatives of the γ's with respect to the time enter the equation (3, 5) shows that the powers of λ in successive terms of the expansions of the γ's may differ by two.[6] We are thus led to the simple assumption that

$$\begin{aligned} \gamma_{00} &= \lambda^2 \underset{2}{\gamma_{00}} + \lambda^4 \underset{4}{\gamma_{00}} + \lambda^6 \underset{6}{\gamma_{00}} + \cdots , \\ \gamma_{0n} &= \lambda^3 \underset{3}{\gamma_{0n}} + \lambda^5 \underset{5}{\gamma_{0n}} + \cdots , \\ \gamma_{mn} &= \lambda^4 \underset{4}{\gamma_{mn}} + \lambda^6 \underset{6}{\gamma_{mn}} + \cdots . \end{aligned} \tag{3, 12}$$

[6] The omission of terms with λ^{2l+1} in γ_{00}, γ_{mn} and with λ^{2l} in γ_{0m} is possible and natural, but logically not strictly necessary. The addition of the omitted terms of (3, 12) could be made in such a way that it would correspond to an introduction of a retarded potential (outgoing wave). Such a procedure would however, be artificial though it would not influence the equations of motion derived in II, as will be shown elsewhere.

We cannot discuss the question of convergence in general, but it is of interest to show that the new method of approximation can give convergent results even where this would not at first be expected. We consider the case of the one-dimensional wave equation in its simplest form

$$f_{xx} - f_{tt} = 0. \tag{3, 13}$$

If, in accordance with the main idea of the new method of approximation, we write

$$\begin{aligned} f &= \underset{0}{f} + \lambda^2 \underset{2}{f} + \lambda^4 \underset{4}{f} + \cdots, \\ f_{xx} &= \underset{0}{f_{xx}} + \lambda^2 \underset{2}{f_{xx}} + \lambda^4 \underset{4}{f_{xx}} + \cdots, \\ f_{tt} &= \lambda^2 f_{\tau\tau} = \lambda^2 \underset{0}{f_{\tau\tau}} + \lambda^4 \underset{2}{f_{\tau\tau}} + \lambda^6 \underset{4}{f_{\tau\tau}} + \cdots, \end{aligned} \tag{3, 14}$$

we obtain from (3, 13) the successive equations

$$\underset{0}{f_{xx}} = 0, \tag{3, 15a}$$

$$\underset{2}{f_{xx}} - \underset{0}{f_{\tau\tau}} = 0, \tag{3, 15b}$$

$$\underset{4}{f_{xx}} - \underset{2}{f_{\tau\tau}} = 0, \tag{3, 15c}$$

From these equations we can find the general solution of the wave equation (3, 13) expressed as a power series in λ. For simplicity we shall consider only the case of a sinusoidal wave so that, out of the totality of solutions of (3, 15a),

$$\underset{0}{f} = A(\tau) + xB(\tau), \tag{3, 16}$$

we choose the particular solution[7]

$$\underset{0}{f} = \sin \tau \tag{3, 17a}$$

and at each subsequent stage of the procedure we ignore all arbitrary functions which may enter. From (3, 15b), (3, 15c), . . . , we thus find

$$\underset{2}{f} = -\frac{x^2}{2!} \sin \tau, \tag{3, 17b}$$

$$\underset{4}{f} = \frac{x^4}{4!} \sin \tau, \tag{3, 17c}$$

so that the solution takes the form

$$f = \sin \tau \left\{ 1 - \frac{(x\lambda)^2}{2!} + \frac{(x\lambda)^4}{4!} - \cdots \right\} = \cos (\lambda x) \sin \tau.$$

[7] The inclusion of the solution $\underset{0}{f} = x \sin \tau$ also leads to sinusoidal waves, as is easily seen.

On replacing τ by λt we have

(3, 18) $$f = \cos(\lambda x) \sin(\lambda t)$$

which is an exact solution of (3, 13).

4. **Expansion Properties of Field Quantities.** We shall show in this section that there is a simple general rule concerning the types of expansion which will occur when we treat the gravitational equations by the present method of approximation. This rule is that

Any component having an odd number of zero suffixes will have only odd powers of λ in its expansion, while any component having an even number of such suffixes will involve only even powers of λ in its expansion.

The fundamental equations (3, 12) show that the $\gamma_{\mu\nu}$ conform to this rule. The relations (1, 19), (1, 20), (1, 21) between $\gamma_{\mu\nu}$ and $h_{\mu\nu}$ have inverse relations of precisely the same form with γ and h interchanged, as

(4, 1) $$h_{00} = \tfrac{1}{2}\gamma_{00} + \tfrac{1}{2}\gamma_{ll}\,,$$

(4, 2) $$h_{0n} = \gamma_{0n}\,,$$

(4, 3) $$h_{mn} = \gamma_{mn} - \tfrac{1}{2}\delta_{mn}\gamma_{ll} + \tfrac{1}{2}\delta_{mn}\gamma_{00}\,,$$

and from (3, 12) it follows that the expansions for the h's in powers of λ are of the form

(4, 4) $$\begin{aligned} h_{00} &= \lambda^2 \underset{2}{h_{00}} + \lambda^4 \underset{4}{h_{00}} + \lambda^6 \underset{6}{h_{00}} + \cdots, \\ h_{0n} &= \lambda^3 \underset{3}{h_{0n}} + \lambda^5 \underset{5}{h_{0n}} + \cdots, \\ h_{mn} &= \lambda^2 \underset{2}{h_{mn}} + \lambda^4 \underset{4}{h_{mn}} + \lambda^6 \underset{6}{h_{mn}} + \cdots, \end{aligned}$$

showing that the h's also conform to the general rule.

Further, since the $\eta_{\mu\nu}$ trivially conform because η_{0n} vanishes, it follows from (1, 4) that the $g_{\mu\nu}$ also conform.

We may write the relation

(4, 5) $$g_{\mu\nu} g^{\nu\sigma} = \delta_\mu^\sigma$$

in the form

(4, 6) $$g_{\mu n} g^{n\sigma} + g_{\mu 0} g^{0\sigma} = \delta_\mu^\sigma.$$

The two groups of terms on the left differ by an even number of zero suffixes so that, since the δ_μ^σ trivially conform to the general rule, we shall obtain enough equations at each approximation for finding the expansions of the $g^{\mu\nu}$ if we assume that the general rule is valid for these components too. However, the $g^{\mu\nu}$ are uniquely determined in terms of the $g_{\mu\nu}$ by (4, 5) so that the expansions according to the general rule will give the only solution and extraneous powers of λ will necessarily have zero coefficients. Thus the rule is applicable to the $g^{\mu\nu}$ and so, also, to the $h^{\mu\nu}$.

Let us consider next the Christoffel symbols of both kinds. We have

$$[\mu\nu, \sigma] = \tfrac{1}{2}(g_{\nu\sigma|\mu} + g_{\sigma\mu|\nu} - g_{\mu\nu|\sigma}), \tag{4, 7}$$

and since the operation "$_{|0}$" introduces a factor λ while the operations "$_{|m}$" leave the order of magnitude unchanged it is evident that the fact that the $g_{\mu\nu}$ obey the general rule implies that the $[\mu\nu, \sigma]$ do too.

The Christoffel symbols of the second kind are defined by

$$\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix} = g^{\lambda\sigma}[\mu\nu, \sigma] \tag{4, 8}$$

and since whenever we have a dummy suffix we shall have either no extra zero suffixes or two such suffixes entering any term in the implied summation, the fact that $g^{\lambda\sigma}$ and $[\mu\nu, \sigma]$ separately conform to the general rule shows that this is true also of the $\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}$.

In the course of the above considerations we have shown that neither the entry of dummy suffixes nor the operations "$_{|m}$", "$_{|0}$" disturbs the operation of the general rule. It follows that if, by the use of these operations alone, we form new quantities from quantities which conform to the rule these new quantities must also obey the general rule. This has already been exemplified by our discussion of the Christoffel symbols, and since all the quantities we shall have to consider, such as

$$R(= g^{\mu\nu}R_{\mu\nu}), \quad \Lambda_{\mu\nu}, \text{ etc.}$$

are new quantities of this type, we see that all the quantities with which we have to deal will have expansions in powers of λ whose general character is summed up in the statement at the head of this section.

5. **Alternative Form of the Equations When Singularities Are Absent.** In this section and the next we shall discuss the case where no singularities are present in the field. This case is, of course, trivial from the physical point of view since it corresponds to the complete absence of matter and, indeed, according to our method of approximation leads to the Galilean solution. Despite this, the discussion of this case will not be without value, for it will serve to exhibit the general mechanism of the theory and will form a convenient introduction to the later, more difficult discussion necessary when singularities are present.

Let us summarize some of the results we have obtained so far. The field has been subjected to the two restrictions

I The Gravitational Field Equations, and

II The Coördinate Conditions,

from which we have found

III The Surface Integral Conditions.

That is, if we take coördinate conditions

(1, 25) $$\gamma_{00|0} - \gamma_{0n|n} = 0,$$

(1, 26) $$\gamma_{nm|n} = 0,$$

the field equations take the form

(1, 30) $$\gamma_{00|ss} = 2\Lambda_{00},$$

(1, 31) $$\gamma_{0n|ss} = 2\Lambda_{0n},$$

(1, 32) $$\gamma_{mn|ss} = 2\Lambda_{mn},$$

and from these two groups of equations we obtain the surface integral conditions

(2, 7) $$\int (\gamma_{00|0n} - 2\Lambda_{0n}) \cos (\mathbf{n}\cdot\mathbf{N})\, dS = 0,$$

(2, 8) $$\int 2\Lambda_{mn} \cos (\mathbf{n}\cdot\mathbf{N})\, dS = 0,$$

and also the results

(2, 9) $$(\gamma_{00|0n} - 2\Lambda_{0n})_{|n} = 2\Lambda_{00|0} - 2\Lambda_{0n|n} = 0,$$

(2, 10) $$2\Lambda_{mn|n} = 0,$$

which are essential for the validity of the surface integral conditions for arbitrary surfaces.

We shall now show that the following two sets of equations (5, 1), (5, 2) are equivalent when no singularities are present.

(5, 1)	(5, 2)
(a) $\gamma_{00\|ss} = 2\Lambda_{00}$,	(a) $\gamma_{00\|ss} = 2\Lambda_{00}$,
(b) $\gamma_{0n\|ss} = 2\Lambda_{0n}$,	(b) $\gamma_{0n\|ss} = 2\Lambda_{0n}$,
(c) $\gamma_{00\|0} - \gamma_{0n\|n} = 0$;	(c′) $\Lambda_{00\|0} - \Lambda_{0n\|n} = 0$,
	(c″) $\int (\gamma_{00\|0n} - 2\Lambda_{0n}) \cos (\mathbf{n}\cdot\mathbf{N})\, dS = 0$;
(d) $\gamma_{mn\|ss} = 2\Lambda_{mn}$,	(d) $\gamma_{mn\|ss} = 2\Lambda_{mn}$,
(e) $\gamma_{mn\|n} = 0$,	(e′) $\Lambda_{mn\|n} = 0$,
	(e″) $\int 2\Lambda_{mn} \cos (\mathbf{n}\cdot\mathbf{N})\, dS = 0$.

In (5, 1) we have merely the field equations and coördinate conditions and we show essentially that the coördinate conditions may be replaced by the surface integral conditions[8] and the conditions (2, 9) (2, 10). The proof for the present

[8] When singularities are absent (5, 2c′), (5, 2c″) and also (5, 2e′), (5, 2e″) are equivalent equations, but we include them all here in order to facilitate comparison with the situation which arises when singularities are present.

case is trivial. For we have already shown that (5, 1) implies (5, 2) and the converse follows at once from the following considerations.

From (5, 2a) (5, 2b) and (5, 2c′) we have

$$(\gamma_{00|0} - \gamma_{0n|n})_{|ss} = 2\Lambda_{00|0} - 2\Lambda_{0n|n} = 0,$$

and since there are no singularities and the γ's must be zero at infinity this gives

$$\gamma_{00|0} - \gamma_{0n|n} = 0$$

which is (5, 1c). The proof for γ_{mn} is similar.

6. **Splitting of the Equations When Singularities Are Absent.** In the first section we gave a prescription for separating the terms of each of the field equations into two well-defined groups. In this section we shall discuss the splitting of the gravitational equations according to powers of λ and shall show why just this method of separation is implied by our method of approximation.

It is necessary first to introduce certain notations. Consider the quantity

$$h_{mn|0s}\,. \tag{6, 1}$$

When h_{mn} is expanded in powers of λ we write

$$h_{mn} = \lambda^2 \underset{2}{h_{mn}} + \lambda^4 \underset{4}{h_{mn}} + \cdots + \lambda^{2l} \underset{2l}{h_{mn}} + \cdots, \tag{6, 2}$$

where the numbers underneath the h's on the right serve the double purpose of distinguishing between the different functions h on the right and of showing with what power of λ each is associated in the expansion.

Now the fundamental assumption of our method of approximation requires that h_{mn} be a function of (λx^0, x^1, x^2, x^3) so that

$$h_{mn|s} = \frac{\partial h_{mn}}{\partial x^s}$$

but

$$h_{mn|0} = \frac{\partial h_{mn}}{\partial x^0} = \lambda \frac{\partial h_{mn}}{\partial \tau}.$$

In order to distinguish between ordinary differentiation with respect to (x^0, x^1, x^2, x^3) and ordinary differentiation with respect to (τ, x^1, x^2, x^3) we shall denote the latter by a comma followed by an appropriate suffix:

$$h_{mn|s} = \frac{\partial h_{mn}}{\partial x^s} = h_{mn,s}\,, \tag{6, 3}$$

$$h_{mn|0} = \frac{\partial h_{mn}}{\partial x^0} = \lambda \frac{\partial h_{mn}}{\partial \tau} = \lambda\, h_{mn,0}\,. \tag{6, 4}$$

Thus h_{mn}, $h_{mn,s}$ and $h_{mn,0}$ are all of the same order, but $h_{mn|0}$ belongs to a power of λ one higher.

With this convention we may write the expansion of (6, 1) in the form

$$(6, 5) \qquad h_{mn|0s} = \lambda h_{mn,0s} = \lambda^3 \underset{2}{h}_{mn,0s} + \lambda^5 \underset{4}{h}_{mn,0s} + \cdots + \lambda^{2l+1} \underset{2l}{h}_{mn,0s} + \cdots.$$

Now, however, the number underneath each h on the right no longer indicates directly the power of λ with which it is associated. We therefore write a 1 underneath each zero suffix following a comma for every h having a number underneath so that (6, 5) becomes

$$(6, 6) \qquad h_{mn|0s} = \lambda h_{mn,0s} = \lambda^3 \underset{2}{h}_{mn,\underset{1}{0}s} + \lambda^5 \underset{4}{h}_{mn,\underset{1}{0}s} + \cdots + \lambda^{2l+1} \underset{2l}{h}_{mn,\underset{1}{0}s} + \cdots.$$

Thus now the sum of the numbers underneath each h gives the power of λ with which it is associated while the first of these numbers indicates the particular function h we are considering. This notation is then consistent with the natural notation for a product of h's.

We consider now what happens when we introduce the power series expansions for the h's in the equations (1, 27), (1, 28), (1, 29). On equating to zero the coefficients of the various powers of λ we shall obtain

$$(6, 7) \qquad \underset{2l}{h}_{00,ss} = 2\underset{2l}{L}_{00},$$

$$(6, 8) \qquad \underset{2l+1}{h}_{0n,ss} = 2\underset{2l+1}{L}_{0n},$$

$$(6, 9) \qquad \underset{2l}{h}_{mn,ss} = 2\underset{2l}{L}_{mn}.$$

The lowest h's are $\underset{2}{h}_{00}$, $\underset{3}{h}_{0n}$, and $\underset{2}{h}_{mn}$, and these will therefore be the quantities determined in the first approximation. They correspond to $l = 1$ in the scheme of (6, 7) (6, 8) (6, 9). Thus at any stage, say l, the quantities to be determined are $\underset{2l}{h}_{00}$, $\underset{2l+1}{h}_{0n}$, $\underset{2l}{h}_{mn}$, and the quantities already known from the solutions of the previous approximations are the h's having lower numbers underneath.

But if we look at the forms of the L's, as given in (1, 15), (1, 16), (1, 17) we see that at the stage l we have either quadratic terms or linear terms involving differentiations with respect to x^0. The quadratic terms can only involve h's of lower order than for l, and the linear terms may be written as

$$(6, 10) \qquad \underset{2l-1}{h}_{0s,\underset{1}{0}s} - \tfrac{1}{2}\underset{2l-2}{h}_{ss,\underset{1}{0}\underset{1}{0}} \qquad \text{in } \underset{2l}{L}_{00},$$

$$(6, 11) \qquad \text{none} \qquad \text{in } \underset{2l+1}{L}_{0n},$$

$$(6, 12) \qquad \tfrac{1}{2}\underset{2l-2}{h}_{mn,\underset{1}{0}\underset{1}{0}} - \tfrac{1}{2}\underset{2l-1}{h}_{0m,\underset{1}{0}n} - \tfrac{1}{2}\underset{2l-1}{h}_{0n,\underset{1}{0}m} \qquad \text{in } \underset{2l}{L}_{mn}.$$

These are all known functions from the previous approximations. Thus the whole of $\underset{2l}{L_{00}}$, $\underset{2l+1}{L_{0n}}$, $\underset{2l}{L_{mn}}$ for given l are known from the solutions of the previous approximations. This is the reason for the particular method of separation of the field equations into two parts described in 1. When the separation is made in this manner and the power series expansions are inserted for the h's in (1, 27) (1, 28), (1, 29), for each power of λ the corresponding coefficients automatically group themselves into those quantities which enter for the first time with the approximation in question and those which are already known, at least in principle, from the previous approximations. These two groups correspond exactly to the left and right hand sides of (1, 27), (1, 28), (1, 29).

Before we can solve the approximation equations we must also split the coördinate conditions (1, 25), (1, 26), and the relations between the h's and γ's according to powers of λ. It turns out that we may take at each stage

$$\underset{2l}{\gamma_{00,ss}} = 2\underset{2l}{\Lambda_{00}}, \qquad \underset{2l+1}{\gamma_{0n,ss}} = 2\underset{2l+1}{\Lambda_{0n}}, \qquad \underset{2l}{\gamma_{00}}{}_{,\underset{1}{0}} - \underset{2l+1}{\gamma_{0n,n}} = 0; \tag{6, 13}$$

$$\underset{2l}{\gamma_{mn,ss}} = 2\underset{2l}{\Lambda_{mn}}, \qquad \underset{2l}{\gamma_{mn,n}} = 0, \tag{6, 14}$$

where the Λ's are known because of the solutions of the previous approximations.

We may also split the alternative equations (5, 2) and use, instead, at each stage

$$\begin{gathered} \underset{2l}{\gamma_{00,ss}} = 2\underset{2l}{\Lambda_{00}}, \qquad \underset{2l+1}{\gamma_{0n,ss}} = 2\underset{2l+1}{\Lambda_{0n}}, \qquad \underset{2l}{\Lambda_{00}}{}_{,\underset{1}{0}} - \underset{2l+1}{\Lambda_{0n,n}} = 0, \\ \int \left(\underset{2l}{\gamma_{00}}{}_{,\underset{1}{0}n} - 2\underset{2l+1}{\Lambda_{0n}} \right) \cos(\mathbf{n}\cdot\mathbf{N})\, dS; \end{gathered} \tag{6, 15}$$

$$\underset{2l}{\gamma_{mn,ss}} = 2\underset{2l}{\Lambda_{mn}}, \qquad \underset{2l}{\Lambda_{mn,n}} = 0, \qquad \int 2\underset{2l}{\Lambda_{mn}} \cos(\mathbf{n}\cdot\mathbf{N})\, dS = 0. \tag{6, 16}$$

As in the case of the unsplit equations, the surface integral conditions are consequences of the others because of the absence of singularities, and the whole splitting actually presents no fundamental difficulties for this case.

7. The General Theory When Singularities Are Present. The existence of singularities in the field introduces certain factors which make the theory developed for the regular case inadequate. For, although the equations of the field are undefined at the singularities, their validity in the regular region is sufficient to determine the motion of these singularities. The slightest alteration in the position of a singularity amounts to an arbitrarily large alteration for a point near enough to the singularity, and we are therefore not permitted to make use of approximate expressions for the equations of motion in the development of our method of approximation. This fact leads to a new difficulty, in the approximation method, which must be discussed more fully.

Let there be p particles producing the field. We may represent their positions at any time by means of their spatial coördinates $\overset{k}{\xi}{}^m(\tau)$, $k = 1, 2, \cdots, p$. At these points the field will be singular, but we may enclose each of the singularities within a small surface,[9] and then the region exterior to these p surfaces will be regular.

Although the equations (5, 1), (5, 2) are undefined at the singularities, they have meaning in the regular region and we shall show that they can still be regarded as in some sense equivalent. The discussion can be divided into two parts, one dealing with the (a), (b) and (c) equations, which involve the suffix zero, and the other with the remaining equations having only spatial suffixes. We consider the latter. The essential structure of the (d) and (e) equations is preserved if we omit the suffix m and write for the total field

(7, 1)		(7, 2)	
(d)	$\gamma_{n\|ss} = 2\Lambda_n,$	(d)	$\gamma_{n\|ss} = 2\Lambda_n,$
(e)	$\gamma_{n\|n} = 0,$	(e′)	$\Lambda_{n\|n} = 0,$
		(e″)	$\int 2\Lambda_n \cos(\mathbf{n}\cdot\mathbf{N})\, dS = 0.$

The proof that (7, 1) implies (7, 2) has already been given in essence in 2. To prove the converse we first obtain from (7, 2d)

$$\gamma_{n|nss} = 2\Lambda_{n|n}, \tag{7, 3}$$

this being valid outside the surfaces enclosing the singularities. To solve this we make an analytic continuation of the functions Λ_n into the interiors of these surfaces in such a way that $\Lambda_{n|n}$ is everywhere zero. This is certainly possible because of the validity of (7, 2e″). So (7, 3) now becomes

$$\gamma_{n|nss} = 0$$

which, being everywhere valid, has the unique solution

$$\gamma_{n|n} = 0$$

which is (7, 1e).

Thus we have shown that if we make an analytic continuation of Λ_n so that (7, 2e′) is valid everywhere, then (7, 1) and (7, 2) are equivalent outside the surfaces enclosing the singularities.

It is clear from the proof that the result will hold for any surfaces enclosing the singularities.

For the (a), (b) and (c) equations a similar proof can also be given. In this case it is necessary to make an analytic continuation of the quantities Λ_{00} and

[9] Throughout the argument we assume that we are dealing with the situation at some definite time τ, allowing time to flow again only after the argument is concluded.

Λ_{0n} in such a way that (5, 2c′) is valid everywhere, this being possible because of (5, 2c″). We omit the details of this part of the proof that (5, 1), (5, 2) may be considered as equivalent even where singularities are present and shall regard the proof as complete.

To show the difficulty brought in by the use of our approximation method let us now consider only the equations (7, 2d), (7, 2e′) omitting the surface integral (7, 2e″). These equations determine the field in each of the approximation steps if the motions of the singularities are prescribed. The motion of the particles is then arbitrary as, for example, in the electrodynamical problem and the field is determined in each of the approximation steps by the equations

$$\underset{2l}{\gamma_{n,ss}} = 2\underset{2l}{\Lambda_n}$$

$$\underset{2l}{\Lambda_{n,n}} = 0.$$

The contradiction is evident if we try to add to these equations the surface condition split according to our approximation method. We then have the additional equation

$$\int^{k} 2\underset{2l}{\Lambda_n} \cos(\mathbf{n}\cdot\mathbf{N}) = 0 \tag{7, 4}$$

where (k) on top of an integral sign means that the surface of integration encloses only the k-singularity. We have in (7, 4) an infinite set of equations containing the functions $\overset{k}{\xi}$ and their time derivatives. These equations cannot be satisfied by the arbitrarily given $\overset{k}{\xi}$ functions characterising the motion.

This also shows how the difficulty can be avoided. We have to consider instead of (5, 1) or (5, 2) a more general set of conditions governing the field which contains those equations as a particular case. Since it is the surface integral conditions which cause the trouble we remove (5, 2c″), (5, 2e″) from the set (5, 2) and consider the significance of what remains.

In making this generalisation we have, of course, gone beyond the gravitational equations to others which contain them as a special case, and we must now discuss what changes have been induced in (5, 1) by this generalisation.

Since the surface integrals are independent of the surfaces, their values will be functions of the time alone through the ξ's and their derivatives. There is therefore no loss of generality if we denote these integrals taken over the p surfaces enclosing the various singularities by $4\pi\overset{k}{c}_0(\tau)$, $4\pi\overset{k}{c}_m(\tau)$:

$$\begin{aligned} \frac{1}{4\pi}\int^{k} (\gamma_{00|0} - 2\Lambda_{0n}) \cos(\mathbf{n}\cdot\mathbf{N})\, dS &= \overset{k}{c}_0(\tau), \\ \frac{1}{4\pi}\int^{k} 2\Lambda_{mn} \cos(\mathbf{n}\cdot\mathbf{N})\, dS &= \overset{k}{c}_m(\tau). \end{aligned} \tag{7, 5}$$

With this notation we shall now prove that the following two sets of equations (7, 6), (7, 7) are equivalent in a certain sense which will be explained in the course of the proof:

(7, 6)		(7, 7)	
(a)	$\gamma_{00\|ss} = 2\Lambda_{00},$	(a)	$\gamma_{00\|ss} = 2\Lambda_{00},$
(b)	$\gamma_{0n\|ss} = 2\Lambda_{0n},$	(b)	$\gamma_{0n\|ss} = 2\Lambda_{0n},$
(c)	$\gamma_{00\|0} - \gamma_{0n\|n} = -\sum_{k=1}^{p} \left\{\overset{k}{c_0}/\overset{k}{r}\right\};$	(c′)	$\Lambda_{00\|0} - \Lambda_{0n\|n} = 0;$
(d)	$\gamma_{mn\|ss} = 2\Lambda_{mn},$	(d)	$\gamma_{mn\|ss} = 2\Lambda_{mn},$
(e)	$\gamma_{mn\|n} = -\sum_{k=1}^{p} \left\{\overset{k}{c_m}/\overset{k}{r}\right\},$	(e′)	$\Lambda_{mn\|n} = 0.$

Here $\overset{k}{r}$ is the "distance" from x^n to the k-singularity:

$$\overset{k}{r} = \left[\left(x^s - \overset{k}{\xi^s}\right)\left(x^s - \overset{k}{\xi^s}\right)\right]^{\frac{1}{2}}. \tag{7, 8}$$

We may introduce the surfaces enclosing the singularities as before and these equations will certainly have meaning outside them. The proof of their equivalence can here too be broken up into two parts and we shall only prove the equivalence for the (d) and (e) parts. Omitting the suffix m as before, we have

(7, 9)		(7, 10)	
(d)	$\gamma_{n\|ss} = 2\Lambda_n,$	(d)	$\gamma_{n\|ss} = 2\Lambda_n,$
(e)	$\gamma_{n\|n} = -\sum_{k=1}^{p} \left\{\overset{k}{c}/\overset{k}{r}\right\},$	(e′)	$\Lambda_{n\|n} = 0,$

with the notation

$$\frac{1}{4\pi} \int^{k} 2\Lambda_n \cos(\mathbf{n} \cdot \mathbf{N})\, dS = \overset{k}{c}(\tau). \tag{7, 11}$$

We begin by proving that (7, 10) implies (7, 9) under certain conditions of continuation. It is no longer possible to make an analytic continuation of Λ_n in such a way that (e′) is everywhere satisfied since this would imply that the surface integral is necessarily zero. In fact, from (7, 11) we see, by Gauss' theorem, that the continuation must be such that

$$\frac{1}{4\pi} \int^{k} 2\Lambda_{n|n}\, dv = \frac{1}{4\pi} \int^{k} 2\Lambda_n \cos(\mathbf{n} \cdot \mathbf{N})\, dS = \overset{k}{c}(\tau). \tag{7, 12}$$

It is simplest for our purposes to make the continuation in such a way that Λ_n and $\Lambda_{n|n}$ are continuous at the surfaces, and that $\Lambda_{n|n}$ has a constant sign inside each surface and satisfies (7, 12).

Such a continuation is possible for any surfaces surrounding the singularities and can be made in such a way that when these surfaces shrink to zero size the function $\Lambda_{n|n}$ goes over to a sum of Dirac δ-functions:

$$\Lambda_{n|n} \to 4\pi \sum_{k=1}^{p} \overset{k}{c}(\tau)\cdot\delta\Big(x^1 - \overset{k}{\xi}{}^1\Big)\,\delta\Big(x^2 - \overset{k}{\xi}{}^2\Big)\,\delta\Big(x^3 - \overset{k}{\xi}{}^3\Big). \tag{7, 13}$$

From (7, 10d) we have now

$$\gamma_{n|nss} = 2\Lambda_{n|n}$$

so that

$$\gamma_{n|n}(x) = -\frac{1}{4\pi}\int \frac{2\Lambda_{n|n}(x')}{r(x, x')}\,dv', \tag{7, 14}$$

where the integral is to be taken over the whole domain of x^n and $r(x, x')$ is the "distance" from x^n to x'^n:

$$r(x, x') = [(x^s - x'^s)(x^s - x'^s)]^{\frac{1}{2}}. \tag{7, 15}$$

Because of the validity of (7, 10e′) outside the surfaces we may write (7, 14) as

$$\gamma_{n|n}(x) = -\frac{1}{4\pi}\sum_{k=1}^{p}\int^{k} \frac{2\Lambda_{n|n}(x')}{r(x, x')}\,dv', \tag{7, 16}$$

the integrals being taken only over the interiors of the surfaces. On shrinking these surfaces we may regard $r(x, x')$ as constants over the various domains of integration and write

$$\gamma_{n|n}(x) = -\frac{1}{4\pi}\sum_{k=1}^{p} 1/\Big(\overset{k}{r}(x)\Big)\int^{k} 2\Lambda_{n|n}\,dv,$$

and by (7, 12) this is

$$\gamma_{n|n} = -\sum_{k=1}^{p}\left\{\overset{k}{c}(\tau)/\overset{k}{r}\right\}$$

which is (7, 9e).

We have therefore shown that with the analytic continuation used above the equations (7, 10) imply the equations (7, 9).

To prove the converse we form from (7, 9) the relation

$$(\gamma_{n|s} - \gamma_{n|n})_{|s} = 2\Lambda_n + \sum_{k=1}^{p}\left\{\overset{k}{c}(\tau)/\overset{k}{r}\right\}_{|n}. \tag{7, 17}$$

If we now form the surface integrals of the "normal" components of the two sides of this equation for each of the surfaces enclosing the singularities in turn, the left hand side will give zero, as explained in 2, and we shall have left

$$\int^{k} 2\Lambda_n \cos(\mathbf{n}\cdot\mathbf{N})\,dS = -\int^{k}\left\{\overset{k}{c}/\overset{k}{r}\right\}_{|n}\cos(\mathbf{n}\cdot\mathbf{N})\,dS$$
$$= -\overset{k}{c}\int^{k}\left\{1/\overset{k}{r}\right\}_{|n}\cos(\mathbf{n}\cdot\mathbf{N})\,dS = 4\pi\overset{k}{c}(\tau)$$

which is (7.11).

That the validity of (7, 10e′) for the regular region is contained in (7, 9) is trivial, and the equivalence of (7, 6d, e) and (7, 7d, e′) is therefore proved.

The proof of the equivalence for the remaining equations of (7, 6), (7, 7) containing the suffix zero presents no essentially new problems and will be omitted.

The whole point of our elaborate procedure in writing all the equations of the field in two equivalent forms is now clear since the present generalisation from (5, 1) to (7, 6) could not be made in a convincing manner without the aid of the parallelism with (5, 2) and (7, 7).

Owing to the absence of the surface integral conditions in (7, 7), there is no longer any objection to the application of our method of approximation to the solution of this set of equations. The λ's will cause a splitting of the equations just as before, except that the surface integral conditions will be absent. However, at each stage we may write

$$(7,\ 18)\qquad \begin{aligned} \frac{1}{4\pi}\int^{k}\Big(\underset{2l\ \ 1}{\gamma_{00,0n}} - \underset{2l+1}{2\Lambda_{0n}}\Big)\cos(\mathbf{n}\cdot\mathbf{N})\,dS &= \underset{2l+1}{\overset{k}{c_0}}(\tau),\\ \frac{1}{4\pi}\int^{k} \underset{2l}{2\Lambda_{mn}}\cos(\mathbf{n}\cdot\mathbf{N})\,dS &= \underset{2l}{\overset{k}{c_m}}(\tau), \end{aligned}$$

and with this notation we have the result, in precisely the same manner as for (7, 6), (7, 7), that for each stage of the approximation the following sets of equations (7, 19), (7, 20) are equivalent:

(7, 19)		(7, 20)	
(a)	$\underset{2l}{\gamma_{00,ss}} = \underset{2l}{2\Lambda_{00}}$,	(a)	$\underset{2l}{\gamma_{00,ss}} = \underset{2l}{2\Lambda_{00}}$,
(b)	$\underset{2l+1}{\gamma_{0n,ss}} = \underset{2l+1}{2\Lambda_{0n}}$,	(b)	$\underset{2l+1}{\gamma_{0n,ss}} = \underset{2l+1}{2\Lambda_{0n}}$,
(c)	$\underset{2l\ \ 1}{\gamma_{00,0}} - \underset{2l+1}{\gamma_{0n,n}} = -\sum_{k=1}^{p}\Big\{\underset{2l+1}{\overset{k}{c_0}}(\tau)/\overset{k}{r}\Big\}$;	(c′)	$\underset{2l\ \ 1}{\Lambda_{00,0}} - \underset{2l+1}{\Lambda_{0n,n}} = 0$;
(d)	$\underset{2l}{\gamma_{mn,ss}} = \underset{2l}{2\Lambda_{mn}}$,	(d)	$\underset{2l}{\gamma_{mn,ss}} = \underset{2l}{2\Lambda_{mn}}$,
(e)	$\underset{2l}{\gamma_{mn,n}} = -\sum_{k=1}^{p}\Big\{\underset{2l}{\overset{k}{c_m}}(\tau)/\overset{k}{r}\Big\}$,	(e′)	$\underset{2l}{\Lambda_{mn,n}} = 0$.

In the actual solving of the equations it is simpler to work with the sets (7, 19) rather than with (7, 20). At each stage we have to solve equations of the type $\gamma_{,ss} = 2\Lambda$ and in order to make the whole solution unambiguous we must impose the conditions that the field shall be Galilean at infinity and that no harmonic functions of higher type than simple poles may be added to the partial solutions except insofar as their addition is forced by the coördinate conditions (7, 19c), (7, 19e). Let us suppose we have been able to solve all the successive approximations. Then the quantities $\overset{k}{c}_0(\tau)$, $\overset{k}{c}_m(\tau)$ are given by the relations

$$\overset{k}{c}_0(\tau) = \sum_{l=1}^{\infty} \lambda^{2l+1} \underset{2l+1}{\overset{k}{c}_0}(\tau), \tag{7, 21}$$

$$\overset{k}{c}_m(\tau) = \sum_{l=1}^{\infty} \lambda^{2l} \underset{2l}{\overset{k}{c}_m}(\tau). \tag{7, 22}$$

Our solution will not in general be a solution of the gravitational equations since (7, 6), (7, 7) are more general than those equations. However, if we now put

$$\overset{k}{c}_0(\tau) = 0, \qquad \overset{k}{c}_m(\tau) = 0 \tag{7, 23}$$

we impose such conditions on the motions of the singularities that our solutions will indeed become the solutions of the gravitational equations we are actually interested in.

The differential equations (7, 23) for the ξ's are really independent of λ since they must be expressed in terms, not of the auxiliary time τ but of the true time x^0, and when this is done the λ's will be necessarily reabsorbed.

In practice, of course, it is impossible to carry the computation beyond the first few stages. Let us suppose, then, that we have been able to solve the successive approximations up to some stage $l = q$. In this case, if we put

$$\sum_{l=1}^{q} \lambda^{2l+1} \underset{2l+1}{\overset{k}{c}_0}(\tau) = 0, \qquad \sum_{l=1}^{q} \lambda^{2l} \underset{2l}{\overset{k}{c}_m}(\tau) = 0, \tag{7, 24}$$

we shall obtain solutions of the gravitational equations correct to terms of the order $(2q + 1)$, and the equations (7, 24) will give the approximate equations of motion up to this order.

8. **The Zero Coördinate Condition.** We show in this section that the solution of our equations can always be made in such a way that

$$\underset{2l+1}{\overset{k}{c}_0}(\tau) = 0, \tag{8, 1}$$

thus showing that the conditions

$$\overset{k}{c}_0(\tau) = \sum_{l=1}^{\infty} \lambda^{2l+1} \underset{2l+1}{\overset{k}{c}_0}(\tau) \tag{7, 21}$$

place no restriction on the motion of the singularities. This result is of significance because the conditions (7, 22) are alone sufficient to describe the motion completely and any further condition, if not redundant, would cause an overdetermination of the motion.

We actually use (8, 1) as normalisation conditions for each stage of the approximation and they are essential for the uniqueness of the solution.

The significant equations for the present argument are

(7, 19a) $$\underset{2l}{\gamma_{00,ss}} = 2\underset{2l}{\Lambda_{00}},$$

(7, 19b) $$\underset{2l+1}{\gamma_{0n,ss}} = 2\underset{2l+1}{\Lambda_{0n}},$$

(7, 19c) $$\underset{2l}{\gamma}{}_{00,\underset{1}{0}} - \underset{2l+1}{\gamma_{0n,n}} = -\sum_{k=1}^{p}\left\{\underset{2l+1}{\overset{k}{c}_0(\tau)}/\overset{k}{r}\right\},$$

where the Λ's are known from the solutions of the previous approximations, and we shall suppose that we have a solution of these equations. If we introduce the quantities $\overset{k}{\Gamma}(\tau)$ by means of the equation

(8, 2) $$\underset{2l}{\overset{k}{\Gamma}}{}_{,\underset{1}{0}}(\tau) = \underset{2l+1}{\overset{k}{c}_0}(\tau),$$

we may write (7, 19c) in the form

(8, 3) $$\begin{aligned}\underset{2l}{\gamma}{}_{00,\underset{1}{0}} - \underset{2l+1}{\gamma_{0n,n}} &= -\sum_{k=1}^{p}\left\{\left(\underset{2l}{\overset{k}{\Gamma}}/\overset{k}{r}\right)_{,\underset{1}{0}} - \underset{2l}{\overset{k}{\Gamma}}\left(1/\overset{k}{r}\right)_{,\underset{1}{0}}\right\}\\ &= -\sum_{k=1}^{p}\left\{\left(\underset{2l}{\overset{k}{\Gamma}}/\overset{k}{r}\right)_{,\underset{1}{0}} + \left(\underset{2l}{\overset{k}{\Gamma}}\,\underset{1}{\overset{k}{\dot{\xi}}}{}^{n}/\overset{k}{r}\right)_{,n}\right\},\end{aligned}$$

where $\dot{\xi} = \dfrac{d\xi}{d\tau}$.

From (7, 19a), (7, 19b) we see that $\underset{2l}{\gamma_{00}}$ and $\underset{2l+1}{\gamma_{0n}}$ are arbitrary to the extent of additive harmonic functions and we may therefore add simple poles to them to form the new quantities

(8, 4) $$\underset{2l}{\gamma'_{00}} = \underset{2l}{\gamma_{00}} + \sum_{k=1}^{p}\left\{\underset{2l}{\overset{k}{\Gamma}}/\overset{k}{r}\right\},$$

(8, 5) $$\underset{2l+1}{\gamma'_{0n}} = \underset{2l+1}{\gamma_{0n}} - \sum_{k=1}^{p}\left\{\underset{2l}{\overset{k}{\Gamma}}\,\overset{k}{\dot{\xi}}{}^{n}/\overset{k}{r}\right\}.$$

These new γ's however, while still satisfying (7, 19a), (7, 19b) will be such that

(8, 6) $$\underset{2l}{\gamma'}{}_{00,\underset{1}{0}} - \underset{2l+1}{\gamma'_{0n,n}} = 0.$$

Since the c_0's now vanish, the surface integrals will also be zero and thus the zero coördinate condition will not affect the motion. This theorem and our previous results show us that the equations which must form the basis of the actual calculation of the field and the equations of motion of the singularities are:

(8, 7a)
$$\underset{2l}{\gamma_{00,ss}} = 2\underset{2l}{\Lambda_{00}},$$

(8, 7b)
$$\underset{2l+1}{\gamma_{0n,ss}} = 2\underset{2l+1}{\Lambda_{0n}},$$

(8, 7c)
$$\underset{2l}{\gamma_{00,}}\underset{1}{_{0}} - \underset{2l+1}{\gamma_{0n,n}} = 0;$$

(8, 7d)
$$\underset{2l}{\gamma_{mn,ss}} = 2\underset{2l}{\Lambda_{mn}},$$

and

(8, 7e)
$$\underset{2l}{\gamma_{mn,n}} = -\sum_{k=1}^{p}\left\{\underset{2l}{\overset{k}{c}_m}(\tau)/\overset{k}{r}\right\},$$

with

(8, 8)
$$\underset{2l}{\overset{k}{c}_m}(\tau) = \frac{1}{4\pi}\int^{k} 2\underset{2l}{\Lambda_{mn}} \cos(\mathbf{n}\cdot\mathbf{N})\, dS.$$

The approximate equations of motion for the stage $l = q$ are given by

(8, 9)
$$\sum_{l=1}^{q} \lambda^{2l}\underset{2l}{\overset{k}{c}_m}(\tau) = 0.$$

II. Application of the General Theory

Note. In the first part of this paper we developed the general theory of a new method for solving the equations of gravitation by successive approximation and for obtaining the equations of motion, in principle to any desired degree of accuracy. In the present part we deal with the actual application of this method, carrying the calculation to such a stage that the main deviation from the Newtonian laws of motion is determined.

Unfortunately, as the work proceeds, the calculations become more and more extensive involving a great amount of technical detail which can have no intrinsic interest. To give all these calculations explicitly here would be quite impracticable and we are obliged to confine ourselves to stressing the general ideas of the work and merely announcing the actual results. For the convenience of anyone who may be interested in the details of the calculation, however, the entire computation of this part of our paper has been deposited with the Institute for Advanced Study so as to be available for reference.[10]

9. **The Approximation $l = 1$.** The approximation $l = 0$ is trivial, leading to the Galilean case, and we proceed at once to the next approximation $l = 1$.

[10] c/o Secretary of the School of Mathematics, Institute for Advanced Study, Princeton, N. J. (U. S. A.).

Since the quantities $\underset{2}{\Lambda}_{00}$, $\underset{3}{\Lambda}_{0n}$, and, $\underset{2}{\Lambda}_{mn}$ are all zero and, as explained in 3, the $\underset{2}{\gamma}_{mn}$ are also zero, we have left from all the equations (8.7a), . . . , (8.7e), merely

$$\underset{2}{\gamma}_{00,ss} = 0, \tag{9.1}$$

$$\underset{3}{\gamma}_{0n,ss} = 0, \tag{9.2}$$

$$\underset{2}{\gamma}_{00,\underset{1}{0}} - \underset{3}{\gamma}_{0n,n} = 0. \tag{9.3}$$

The character of our whole solution will depend essentially upon the choice of the harmonic function we take as the solution of (9.1). We shall assume that the particles we are interested in have spherical symmetry and that the field is Galilean at infinity. In this case the solution of (9.1) is unique since each singularity in $\underset{2}{\gamma}_{00}$ must now, by (9.1) be a simple pole. We therefore have for $\underset{2}{\gamma}_{00}$ the solution

$$\underset{2}{\gamma}_{00} = 2\varphi, \qquad \varphi = \sum_{k=1}^{p}\left\{-2\overset{k}{m}/\overset{k}{r}\right\}, \qquad \overset{k}{r} = \left[\left(x^s - \overset{k}{\xi^s}\right)\left(x^s - \overset{k}{\xi^s}\right)\right]^{\frac{1}{2}}, \tag{9.4}$$

where the p quantities $\overset{k}{m}$ are independent of the spatial coördinates x^s, and can depend at most only on the time.

From (9.2) we see that $\underset{3}{\gamma}_{0n}$ is also a harmonic function, and to determine it more exactly we must use the coördinate (9.3). From (9.3), (9.4) we have

$$\begin{aligned}\underset{3}{\gamma}_{0n,n} = \underset{2}{\gamma}_{00,\underset{1}{0}} &= \sum_{k=1}^{p}\left(-4\overset{k}{m}/\overset{k}{r}\right)_{,0} \\ &= \sum_{k=1}^{p}\left\{\left(4\overset{k}{m}/\overset{k}{r}\right)\dot{\overset{k}{\xi^n}}\right\}_{,n} - \sum_{k=1}^{p}\left(-4\dot{\overset{k}{m}}/\overset{k}{r}\right).\end{aligned}$$

This equation can be solved without introducing new singularities only if $\dot{\overset{k}{m}} = 0$. In other words, the quantities $\overset{k}{m}$, which actually measure the masses of the point singularities, are necessarily constants. It is now evident that, under our general restricting conditions, $\underset{3}{\gamma}_{0n}$ is uniquely determined:

$$\underset{3}{\gamma}_{0n} = \sum_{k=1}^{p}\left\{\left(4\overset{k}{m}/\overset{k}{r}\right)\dot{\overset{k}{\xi^n}}\right\}. \tag{9.5}$$

In all that follows we shall limit our considerations to the case of only two particles. This places no essential restriction on the results as far as the end of 15, their generalisation to p particles being trivial, and it permits a useful simplification of the rather inconvenient notation used for the general case.

For the case of two particles we shall write:

$$(9.6)\qquad \begin{aligned} &(a)\quad -2\overset{1}{m}/\overset{1}{r} = \psi, \qquad -2\overset{2}{m}/\overset{2}{r} = \chi; \\ &(b)\quad \varphi = \psi + \chi; \\ &(c)\quad \overset{1}{\xi^s} = \eta^s, \qquad \overset{2}{\xi^s} = \zeta^s. \end{aligned}$$

Our results (9.4), (9.5) may thus be written in the form

$$(9.7)\qquad \begin{aligned} &(a)\quad \underset{2}{\gamma_{00}} = 2\varphi = 2\psi + 2\chi, \\ &(b)\quad \underset{3}{\gamma_{0n}} = -2\psi\dot{\eta}^n - 2\chi\dot{\zeta}^n. \end{aligned}$$

From (1.18) we now also have

$$(9.8)\qquad \begin{aligned} &(a)\quad \underset{2}{h_{00}} = \varphi = \psi + \chi, \\ &(b)\quad \underset{3}{h_{0n}} = -2\psi\dot{\eta}^n - 2\chi\dot{\zeta}^n, \\ &(c)\quad \underset{2}{h_{mn}} = \delta_{mn}\varphi = \delta_{mn}(\psi + \chi). \end{aligned}$$

This shows that the approximation $l = 1$ has a Newtonian character but, owing to the vanishing of $\underset{2}{\overset{k}{c}_m}$, places no restriction on the motion.

10. **Calculation of the Λ's for $l = 2$.** The first step in the calculation of the Λ's for $l = 2$ is the determination of the $h_{\mu\nu}$.

Using the method explained in 4, we can calculate the expansions of the $h^{\mu\nu}$ to any desired degree of approximation. We find, for $l = 1$,

$$(10.1)\qquad \underset{2}{h^{00}} = -\underset{2}{h_{00}} = -\varphi,$$

$$(10.2)\qquad \underset{3}{h^{0n}} = \underset{3}{h_{0n}} = \underset{3}{\gamma_{0n}},$$

$$(10.3)\qquad \underset{2}{h^{mn}} = -\underset{2}{h_{mn}} = -\delta_{mn}\varphi.$$

We next have to calculate for $l = 2$ the quantities $2L_{\mu\nu}$ defined in (1.15), (1.16), (1.17).

In $2\underset{4}{L_{00}}$ the linear terms give

$$\varphi_{,00}\,.$$

Of the non-linear terms, only three can give a contribution. They are

$$-2\left\{\underset{2}{h^{sr}}\left[\underset{2}{00, r}\right]\right\}_{,s} = -\varphi_{,s}\varphi_{,s} \quad (\text{since } \varphi_{,ss} = 0),$$

$$-2\left[\underset{2}{00, s}\right]\left[\underset{2}{0s, 0}\right] = \tfrac{1}{2}\varphi_{,s}\varphi_{,s}\,,$$

$$-2\left[\underset{2}{00, r}\right]\left[\underset{2}{rs, s}\right] = \tfrac{3}{2}\varphi_{,s}\varphi_{,s}\,,$$

where $[rs, p]$ are Christoffel symbols. Thus

$$2\underset{4}{L}_{00} = \varphi_{,00} + \varphi_{,s}\varphi_{,s}\,. \tag{10.4}$$

Similar but rather more tiresome calculations lead to the further results

$$2\underset{5}{L}_{0n} = \varphi_{,s}\underset{3}{h}_{0s,n} - \varphi_{,sn}\underset{3}{h}_{0s} - 3\varphi_{,0}\varphi_{,n}\,, \tag{10.5}$$

$$2\underset{4}{L}_{mn} = -\underset{3}{h}_{0m,0n} - \underset{3}{h}_{0n,0m} + \delta_{mn}\varphi_{,00} - 2\varphi\varphi_{,mn} - \varphi_{,m}\varphi_{,n} - \delta_{mn}\varphi_{,s}\varphi_{,s}\,. \tag{10.6}$$

Therefore, by (1.30), $\cdots$, (1.35), we have

$$\begin{aligned} &\text{(a)} \quad \underset{4}{\gamma}_{00,ss} = 2\underset{4}{\Lambda}_{00} = -\tfrac{3}{2}\varphi_{,s}\varphi_{,s}\,, \\ &\text{(b)} \quad \underset{5}{\gamma}_{0n,ss} = 2\underset{5}{\Lambda}_{0n} = \varphi_{,s}\underset{3}{\gamma}_{0s,n} - \varphi_{,sn}\underset{3}{\gamma}_{0s} - 3\varphi_{,0}\varphi_{,n}\,, \\ &\text{(c)} \quad \underset{4}{\gamma}_{mn,ss} = 2\underset{4}{\Lambda}_{mn} = -\underset{3}{\gamma}_{0m,0n} - \underset{3}{\gamma}_{0n,0m} + 2\delta_{mn}\varphi_{,00} - 2\varphi\varphi_{,mn} - \varphi_{,m}\varphi_{,n} \\ &\qquad\qquad\qquad + \tfrac{3}{2}\delta_{mn}\varphi_{,s}\varphi_{,s}\,. \end{aligned} \tag{10.7}$$

As explained in 7, 8, these equations (10.7), together with the corresponding coördinate conditions

$$\begin{aligned} &\text{(a)} \quad \underset{4}{\gamma}_{00,\underset{1}{0}} - \underset{5}{\gamma}_{0n,n} = 0, \\ &\text{(b)} \quad \underset{4}{\gamma}_{mn,n} = -\left(\overset{1}{\underset{4}{c}}_m/\overset{1}{r}\right) - \left(\overset{2}{\underset{4}{c}}_m/\overset{2}{r}\right), \end{aligned} \tag{10.8}$$

are the equations which determine the field in the next approximation.

11. The Newtonian Equations of Motion. We must now evaluate the surface integrals

$$\overset{k}{\underset{4}{c}}_m(\tau) = \frac{1}{4\pi}\int^k 2\underset{4}{\Lambda}_{mn}\cos(\mathbf{n}\cdot\mathbf{N})\,dS, \qquad k = 1, 2. \tag{11.1}$$

According to the general theory of part I, these integrals will be independent of the particular shapes of the surfaces of integration since the divergences of their integrands must vanish on a consequence of the field equations belonging to the previous approximation. We shall show here by actual calculation that this is the case with the $2\underset{4}{\Lambda}_{mn}$ given in (10.7c).

Since φ and $\underset{3}{\gamma}_{0n}$ are harmonic functions, we have

$$2\underset{4}{\Lambda}_{mn,n} = -\underset{3}{\gamma}_{0n,0mn} + 2\varphi_{,00m}$$

which is zero, as can easily be seen from (9.3) and (9.7a).

In the actual calculation of the surface integrals we evaluate the separate contributions of the different terms in $2\underset{4}{\Lambda}_{mn}$. Since the value of a whole integral is independent of the shape of its surface of integration, by taking this surface to be of finite size and always a finite distance from its singularity, we see that the whole integral cannot be infinite. Now the individual terms of $2\underset{4}{\Lambda}_{mn}$ have not the property that their divergences vanish, and so we must fix the surfaces of integration quite definitely before we begin the calculations. It is most convenient to take definite, infinitesimally small spheres whose centers are at the singularities, but in this case infinities of the types

$$\lim_{r\to 0} \text{const.}/r^n, \qquad n \text{ a positive integer,}$$

can occur in the values of the partial integrals. Since these must cancel, however, in the final result, we may merely ignore them throughout the calculation of the surface integrals.

We shall consider the integral taken around the first singularity. Owing to the infinitesimal size of the surface of integration, the only terms which can give results different from zero or infinity are those of the order of $\left(1/\overset{1}{r}{}^2\right)$.

The first term in $2\underset{4}{\Lambda}_{mn}$ is $-\underset{3}{\gamma}_{0m,0n}$, and, by (8.7b), this may be written as

$$-\underset{3}{\gamma}_{0m,0n} = -2\psi_{,ns}\dot{\eta}^m\dot{\eta}^s + 2\psi_{,n}\ddot{\eta}^m - 2\chi_{,ns}\dot{\zeta}^m\dot{\zeta}^s + 2\chi_{,n}\ddot{\zeta}^m.$$

The only term we need consider is the second, and so we have

$$\frac{1}{4\pi}\int^1 \left(-\underset{3}{\gamma}_{0m,0n}\right)\cos(\mathbf{n}\cdot\mathbf{N})\,dS = \frac{1}{4\pi}\int^1 2\psi_{,n}\ddot{\eta}^m\cos(\mathbf{n}\cdot\mathbf{N})\,dS$$

$$= \left(4\overset{1}{m}\,\ddot{\eta}^m\right)\frac{1}{4\pi}\int^1\left\{(x^n-\eta^n)(x^n-\eta^n)/\overset{1}{r}{}^4\right\}dS \tag{11.2}$$

$$= \left(4\overset{1}{m}\,\ddot{\eta}^m\right)\frac{1}{4\pi}\int^1\left\{1/\overset{1}{r}{}^2\right\}dS = 4\overset{1}{m}\ddot{\eta}^m.$$

In a similar manner we find that

$$\frac{1}{4\pi}\int^1\left(-\underset{3}{\gamma}_{0n,0m}\right)\cos(\mathbf{n}\cdot\mathbf{N})\,dS = \tfrac{4}{3}\overset{1}{m}\ddot{\eta}^m. \tag{11.3}$$

The fourth term, $(-2\varphi,\ \varphi_{,mn})$, requires slightly different treatment. The only part that can be of interest is

$$-2\psi_{,mn}\,\chi$$

and in order to evaluate the corresponding contribution to the surface integral we must expand χ as a power series in the neighborhood of the first singularity, writing

$$\chi = \tilde{\chi} + (x^s - \eta^s)\tilde{\chi}_{,s} + \cdots, \tag{11.4}$$

where

$$\tilde{\chi} = \chi(\eta^n), \qquad \tilde{\chi}_{,s} = \chi_{,s}(\eta^n), \text{ etc.} \tag{11.5}$$

Introducing this expansion for χ we see that the only term in the integrand which can give a finite result is

$$-2\psi_{,mn}(x^s - \eta^s)\tilde{\chi}_{,s} \tag{11.6}$$

The determination of the surface integral of this term depends on the calculation of

$$\int^1 (x^s - \eta^s)\, \psi_{,mn} \cos(\mathbf{n}\cdot\mathbf{N})\, dS.$$

We have

$$\begin{aligned}(x^s - \eta^s)\, \psi_{,mn} \cos(\mathbf{n}/\mathbf{N}) \\ &= 2\overset{1}{m}(x^s - \eta^s)\left\{-3(x^m - \eta^m)(x^n - \eta^n)/\overset{1}{r}{}^5 + \delta_{mn}/\overset{1}{r}{}^3\right\}(x^n - \eta^n)/\overset{1}{r} \\ &= -4\overset{1}{m}(x^s - \eta^s)(x^m - \eta^m)/\overset{1}{r}{}^4.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{4\pi}\int (x^s - \eta^s)\, \psi_{,mn} \cos(\mathbf{n}\cdot\mathbf{N})\, dS &= -\frac{4\overset{1}{m}}{4\pi}\int (x^s - \eta^s)(x^m - \eta^m)/\overset{1}{r}{}^4\, dS \\ &= -\frac{4\overset{1}{m}}{3}\,\delta_{ms},\end{aligned} \tag{11.7}$$

and so the surface integral of the term (11.6) is

$$+\frac{8\overset{1}{m}}{3}\,\tilde{\chi}_{,m}, \tag{11.8}$$

which is thus also the value of the surface integral for the whole of the term $(-\varphi\varphi_{,mn})$.

In a somewhat similar way we obtain, for the surface integrals of the remaining terms, the values

$$\left\{\begin{aligned} 2\delta_{mn}\varphi_{,00} &\rightarrow -\frac{4\overset{1}{m}}{3}\,\ddot{\overset{1}{\eta}}{}^m, \\ -\varphi_{,m}\varphi_{,n} &\rightarrow -\frac{8\overset{1}{m}}{3}\,\tilde{\chi}_{,m}, \\ \tfrac{3}{2}\delta_{mn}\varphi_{,s}\varphi_{,s} &\rightarrow 2\overset{1}{m}\tilde{\chi}_{,m}. \end{aligned}\right. \tag{11.9}$$

Hence we have

$$\overset{1}{\underset{4}{c}}_m(\tau) = \frac{1}{4\pi}\int^1 2\underset{4}{\Lambda}_{mn}\cos(\mathbf{n}\cdot\mathbf{N})\, dS = 4\overset{1}{m}\{\ddot{\overset{1}{\eta}}{}^m + \tfrac{1}{2}\tilde{\chi}_{,m}\}. \tag{11.10}$$

Let us assume for the moment that we are not going any further with the approximation. In this case our approximate equations of motion would be of the form

$$\lambda^4\{\ddot{\eta}^m + \tfrac{1}{2}\tilde{\chi}_{,m}\} = 0 \tag{11.11}$$

for each particle. It is of interest to note that this form of the equations of motion is actually independent of the variables x^s. For we have, by (11.5), (9.6),

$$\tilde{\chi}_{,s} = \chi_{,s}(\eta^n), \qquad \chi = -2\overset{2}{m}/\overset{2}{r}. \tag{11.12}$$

For our present argument we may take χ as any function of $\overset{2}{r}$. Equations (11.12) show that to form $\tilde{\chi}_{,s}$ we must first differentiate χ with respect to x^s and then replace x^s by η^s. But the result will be the same if we first replace x^s by η^s and later differentiate with respect either to η^s or to $(-\zeta^s)$. Thus

$$\tilde{\chi}_{,s} = \frac{\partial\chi(r)}{\partial\eta^s} = -\frac{\partial\chi(r)}{\partial\zeta^s}, \tag{11.13}$$

where r denotes the "distance" between η^s and ζ^s:

$$r = [(\eta^s - \zeta^s)(\eta^s - \zeta^s)]^{\frac{1}{2}}. \tag{11.14}$$

We can therefore think of our equations of motion as involving the differentiation of functions depending only on the positions of the singularities, as is characteristic of theories based on the concept of action at a distance.

Writing (11.11) more explicitly in vector notation as

$$\overset{1}{m}\ddot{\eta} = \nabla\left(\overset{1}{m}\overset{2}{m}/r\right), \tag{11.15}$$

we see that (11.11) gives precisely the Newtonian law of motion.[11]

We have therefore obtained the Newtonian equations of motion from the field equations alone, without extra assumption such as was hitherto believed to be necessary and was supplied by the law of geodetic lines, or by a special choice of an energy impulse tensor.

From the above derivation of the Newtonian equations of motion, the general mechanism becomes apparent by which the Lorentz equations for the motion of electric particles can be obtained. In this case we have to consider the gravitational equations in which the Maxwell energy-momentum tensor appears on the right, and also the Maxwell field equations, and treat the whole set of equations by our approximation method. It is necessary, now, to give each singularity an electric charge e in addition to its mass m. We may safely ignore the contribution arising from the products of gravitational potentials in

[11] Equation (11.11) and (11.15) are written in terms of the auxiliary time and the auxiliary masses. We shall return to this point in 17.

the new field equations. For this omission has the effect of destroying the second term of (11.11), while the inclusion of the Maxwell tensor leads to the appearance, on the right of (11.11), of a corresponding surface integral giving the electrostatic force acting on the particle. In the next approximation we obtain the full Lorentz force together with the relativistic correction to the mass.

So long as we are dealing with singularities, we have no basis within the theory for excluding negative masses; in other words, for excluding gravitational repulsions between particles. If, however, we decide always to take mass positive, then the sign with which the Maxwell energy-momentum tensor enters the field equations determines whether like charges shall attract or repel each other. This also reveals the limitations of any theory based upon the existence of singularities.

12. **Normalisation of** $\underset{4}{\gamma_{00}}$. The value of $\underset{4}{\gamma_{00}}$ determined from (10.7a) is arbitrary to within an added harmonic function, and this function is to be determined from the relations (8.4), (8.2), together with our basic requirement that higher harmonic functions than simple poles are, as far as possible, to be avoided.

From (10.7a) and the fact that φ is harmonic, we have at once

$$\underset{4}{\gamma_{00}} = -\tfrac{3}{4}\varphi\varphi + \alpha_{00}\psi + \beta_{00}\chi, \tag{12.1}$$

where we have written the additive functions of (8.4) in a different form more in accordance with our present notation, α_{00}, β_{00} being functions of τ alone through η and ζ and their derivatives. The quantities α_{00}, β_{00} can be determined from the condition that

$$\frac{1}{4\pi}\int\left\{\underset{4}{\gamma}{}_{\underset{1}{00,0n}} - 2\underset{5}{\Lambda_{0n}}\right\}\cos\,(\mathbf{n}\cdot\mathbf{N})\,dS = 0. \tag{12.2}$$

The value of α_{00} is found by taking this integral over a small sphere having its center at the first singularity, and from calculations similar to those of 3, we find, after making use of the equations of motion of the first order:

$$\alpha_{00} = \{\dot{\eta}^s\dot{\eta}^s + \tfrac{1}{2}\tilde{\chi}\}. \tag{12.3}$$

Similarly, by integrating over a small sphere around the second singularity, we find

$$\beta_{00} = \{\dot{\zeta}^s\dot{\zeta}^s + \tfrac{1}{2}\tilde{\psi}\}, \tag{12.4}$$

where

$$\begin{aligned}\tilde{\chi} &= \chi(\eta^s),\\ \tilde{\psi} &= \psi(\zeta^s).\end{aligned} \tag{12.5}$$

These results show clearly the physical significance of the particular normalisation required by the conditions (8.4), (8.2). For we now have

$$\lambda^2 \underset{2}{\gamma_{00}} + \lambda^4 \underset{4}{\gamma_{00}} = \lambda^2 \{(1 + \tfrac{1}{2}\lambda^2 \alpha_{00})\, 2\psi + (1 + \tfrac{1}{2}\lambda^2 \beta_{00})\, 2\chi - \tfrac{3}{4}\lambda^2 \varphi\varphi\}, \tag{12.6}$$

and we see from (12.3), (12.4) that $\overset{1}{m}(1 + \frac{1}{2}\lambda^2 \alpha_{00})$, $m(1 + \frac{1}{2}\lambda^2 \beta_{00})$ involve the first relativistic corrections to the masses.

The calculations up to this stage correspond to those of Droste, De Sitter, and Levi-Civita, cited in the introduction.

13. **Solution of the Field Equations for $l = 2$.** Since our ultimate aim is to determine the equations of motion up to the next approximation, we are interested only in those expressions which give a contribution to the corresponding surface integrals. We shall state dogmatically what is needed for these calculations for the justification of our statement can not be given without exposing the details of our actual calculation.

1. The calculation of $\underset{4}{\gamma_{mn}}$ and $\underset{5}{\gamma_{0m}}$ in the neighborhood of the singularities. We do not need to care in $\underset{4}{\gamma_{mn}}$ about those terms which do not go to infinity if $\overset{1}{r} \to 0$.

2. The calculation of $\underset{4}{\gamma_{rr}}$ in the whole space.

The expression $2\underset{4}{\Lambda_{mn}}$ in (10, 7) can be divided into two parts, one containing the linear terms together with all other terms not involving interactions between the two particles, and the other containing all the interaction terms. We denote these two groups of terms respectively by X_{mn} and Y_{mn}. The integration of the equations

$$\underset{4}{\gamma'_{mn,ss}} = X_{mn} \tag{13.1}$$

presents no difficulties, but the equations

$$\underset{4}{\gamma''_{mn,ss}} = Y_{mn} \tag{13.2}$$

cannot, apparently, be integrated in an elementary manner and we are obliged to introduce a simplification. Since we need to know the values of $\underset{4}{\gamma_{mn}}$ mainly in order to evaluate the surface integrals $\underset{6}{c_m}$ about, say, the first particle, we may introduce power series expansions for χ in the neighborhood of this point and so obtain a solution for $\underset{4}{\gamma_{mn}}$ which is also in the form of such an expansion.

We find, actually, from (13.1), (13.2), the following expressions for $\underset{4}{\gamma}'_{mn}$ and $\underset{4}{\gamma}''_{mn}$:

$$\begin{aligned} \underset{4}{\gamma}'_{mn} = {} & \{\psi[(x^n - \eta^n)\dot{\eta}^m + (x^m - \eta^m)\dot{\eta}^n - \delta_{mn}(x^s - \eta^s)\dot{\eta}^s]\}_{,0} \\ & + \{\chi[(x^n - \zeta^n)\dot{\zeta}^m + (x^m - \zeta^m)\dot{\zeta}^n - \delta_{mn}(x^s - \zeta^s)\dot{\zeta}^s]\}_{,0} \\ & + \tfrac{7}{4}\overset{1}{r}{}^2\psi_{,m}\psi_{,n} + \tfrac{7}{4}\overset{2}{r}{}^2\chi_{,m}\chi_{,n}, \end{aligned} \tag{13.3}$$

and

$$\underset{4}{\gamma}''_{mn} = -\psi_{,m}(x^n - \eta^n)\tilde{\chi}, \tag{13.4}$$

where we have included in (13.4) only those terms which ultimately have importance for the evaluation of the surface integrals $\underset{6}{\overset{1}{c}}_m$.

The value of $\underset{4}{\gamma}_{mn}$ is given by

$$\underset{4}{\gamma}_{mn} = \underset{4}{\gamma}'_{mn} + \underset{4}{\gamma}''_{mn} + \alpha_{mn}\psi, \tag{13.5}$$

where α_{mn} is a function of time to be determined from the coördinate conditions

In a similar way, we may calculate the values of $\underset{5}{\gamma}_{0n}$ in two parts. We find, on including only relevant terms for the surface integrals $\underset{6}{\overset{1}{c}}_m$, in the integrands of which $\underset{5}{\gamma}_{0n}$ enters only linearly,

$$\underset{5}{\gamma}'_{0m} = -\tfrac{7}{4}\overset{1}{r}{}^2\psi_{,m}\psi_{,s}\dot{\eta}^s + \tfrac{3}{4}\psi\psi\dot{\eta}^m, \tag{13.6}$$

$$\begin{aligned} \underset{5}{\gamma}''_{0n} = {} & -\tfrac{3}{2}(x^s - \eta^s)\psi\tilde{\chi}_{,m}\dot{\zeta}^s - (x^m - \eta^m)\psi\tilde{\chi}_{,s}\dot{\eta}^s \\ & + \tfrac{1}{2}\psi_{,m}(x^s - \eta^s)(x^l - \eta^l)\tilde{\chi}_{,l}\dot{\zeta}^s + (x^m - \eta^m)\psi\tilde{\chi}_{,s}\dot{\zeta}^s \\ & + \tfrac{1}{2}(x^s - \eta^s)\psi\tilde{\chi}_{,s}\dot{\zeta}^m + \tfrac{3}{2}(x^s - \eta^s)\psi\tilde{\chi}_{,m}\dot{\eta}^s \\ & + (x^s - \eta^s)\psi_{,m}\tilde{\chi}\dot{\zeta}^s. \end{aligned} \tag{13.7}$$

The value of $\underset{5}{\gamma}_{0n}$ is given by

$$\underset{5}{\gamma}_{0n} = \underset{5}{\gamma}'_{0n} + \underset{5}{\gamma}''_{0n} + \alpha_{0n}\psi, \tag{13.8}$$

where α_{0n} is a function of time to be determined from the normalisation condition.

It remains only to calculate $\underset{4}{\gamma}_{rr}$ in the whole space. From (10.7c) we have

$$\underset{4}{\gamma}_{rr,ss} = 2\varphi_{,00} + \tfrac{7}{2}\varphi_{,s}\varphi_{,s}, \tag{13.9}$$

therefore

$$\underset{4}{\gamma}_{rr} = -2\overset{1}{m}\overset{1}{r}_{,00} - 2\overset{2}{m}\overset{2}{r}_{,00} + \tfrac{7}{4}\varphi^2 + \alpha\psi + \beta\chi \tag{13.10}$$

where α and β are functions of time to be determined in such a way that $\underset{4}{\gamma}_{rr}$ in (13.10) would agree with $\underset{4}{\gamma}_{rr}$ determined from (13.5) near to the singularities.

14. **Determination of α_{mn} and α_{0n}.** In order to find α_{mn}, α_{0n} from the conditions (8.7e), (8.7c) we must make use of the values of $\underset{4}{\overset{1}{c}}_m$ found in 3. The result is up to the desired order

$$\alpha_{mn} = \{2\dot{\eta}^m\dot{\eta}^n + \delta_{mn}\tilde{\chi}\} \tag{14.1}$$

and

$$\alpha_{0n} = -\dot{\eta}^s\dot{\eta}^s\dot{\eta}^n + \tilde{\chi}\dot{\eta}^n - \tilde{\chi}\dot{\zeta}^n. \tag{14.2}$$

Finally from our last remark in 13 follows:

$$\alpha = 2\dot{\eta}^s\dot{\eta}^s + \tfrac{1}{2}\tilde{\chi}; \qquad \beta = 2\dot{\zeta}^s\dot{\zeta}^s + \tfrac{1}{2}\tilde{\psi}. \tag{14.3}$$

15. **Calculation of $\underset{6}{\Lambda}_{mn}$.** In the calculation of $\underset{6}{\Lambda}_{mn}$ for our present purposes, we may assume that $\underset{4}{c}_m$ is zero, as we shall now show.

After we have evaluated the surface integrals $\underset{6}{c}_m$, we may write the approximate equations of motion in the form

$$\lambda^4\underset{4}{c}_m + \lambda^6\underset{6}{c}_m = 0. \tag{15.1}$$

But this shows that when the motion is in accordance with (15.1) the quantities $\lambda^4\underset{4}{c}_m$ and $\lambda^6\underset{6}{c}_m$ will be of the same order of magnitude. It is evident, however, that $\lambda^4\underset{4}{c}_m$ can enter $\lambda^6\underset{6}{\Lambda}_{mn}$ only in combination with a quantity of the type $\lambda^2\underset{2}{\Theta}$. It will therefore enter only in terms which actually belong to the order λ^8 or higher, and since we do not propose to go beyond the order λ^6 in the calculation of the equations of motion, we may neglect all terms in $\underset{6}{\Lambda}_{mn}$ in which $\underset{4}{c}_m$ appears. Even if we make use of this fact, however, the calculations are still quite tedious, and there are actually forty-one different types of term in the expansion of $\underset{6}{\Lambda}_{mn}$. We find:

$$
\begin{aligned}
2\underset{6}{\Lambda}_{mn} = & -\underset{5}{\gamma}_{0m,0n} - \underset{5}{\gamma}_{0n,0m} + \delta_{mn}\underset{4}{\gamma}_{00,00} + \underset{4}{\gamma}_{mn,00} - \varphi\underset{4}{\gamma}_{00,mn} - \varphi\underset{4}{\gamma}_{ss,mn} - \varphi_{,mn}\underset{4}{\gamma}_{00} \\
& - \varphi_{,mn}\underset{4}{\gamma}_{ss} + \varphi_{,ms}\underset{4}{\gamma}_{ns} + \varphi_{,ns}\underset{4}{\gamma}_{ms} - \delta_{mn}\varphi_{,sr}\underset{4}{\gamma}_{sr} - 2\varphi_{,s}\underset{4}{\gamma}_{mn,s} + \varphi_{,s}\underset{4}{\gamma}_{ms,n} \\
& + \varphi_{,s}\underset{4}{\gamma}_{ns,m} - \tfrac{1}{2}\varphi_{,m}\underset{4}{\gamma}_{ss,n} - \tfrac{1}{2}\varphi_{,n}\underset{4}{\gamma}_{ss,m} - \tfrac{1}{2}\varphi_{,n}\underset{4}{\gamma}_{00,m} - \tfrac{1}{2}\varphi_{,m}\underset{4}{\gamma}_{00,n} \\
& + \tfrac{3}{2}\delta_{mn}\varphi_{,s}\underset{4}{\gamma}_{rr,s} + \tfrac{3}{2}\delta_{mn}\varphi_{,s}\underset{4}{\gamma}_{00,s} - \underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0n,ms} - \underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0m,ns} + 2\underset{3}{\gamma}_{0s}\underset{3}{\gamma}_{0s,mn} \\
& + \tfrac{1}{2}\delta_{mn}\underset{3}{\gamma}_{0s,r}\underset{3}{\gamma}_{0r,s} - \tfrac{3}{2}\delta_{mn}\underset{3}{\gamma}_{0s,r}\underset{3}{\gamma}_{0s,r} + \underset{3}{\gamma}_{0s,m}\underset{3}{\gamma}_{0s,n} + \underset{3}{\gamma}_{0m,s}\underset{3}{\gamma}_{0ns} - \varphi_{,0n}\underset{3}{\gamma}_{0m} \\
& - \varphi_{,0m}\underset{3}{\gamma}_{0n} + 2\delta_{mn}\varphi_{,0s}\underset{3}{\gamma}_{0s} - \varphi_{,0}\underset{3}{\gamma}_{0m,n} - \varphi_{,0}\underset{3}{\gamma}_{0n,m} - \varphi_{,n}\underset{3}{\gamma}_{0m,0} - \varphi_{,m}\underset{3}{\gamma}_{0n,0} \\
& + 2\varphi\underset{3}{\gamma}_{0m,0n} + 2\varphi\underset{3}{\gamma}_{0n,0m} - 2\delta_{mn}\varphi\varphi_{,00} + 2\varphi\varphi\varphi_{,mn} - \varphi\varphi_{,m}\varphi_{,n} \\
& + \tfrac{3}{2}\delta_{mn}\varphi\varphi_{,s}\varphi_{,s} + \tfrac{1}{2}\delta_{mn}\varphi_{,0}\varphi_{,0}\,.
\end{aligned}
\tag{15.2}
$$

The condition that $\underset{6}{\Lambda}_{mn,n}$ must be zero affords a valuable test of the correctness of the above formula. We have worked out the divergence of the $\underset{6}{\Lambda}_{mn}$ given in (15.2) and have found that it does indeed vanish.

16. **The Surface integrals for $l = 3$.** In order to find the principal deviation from the Newtonian laws of motion, all that essentially remains is to calculate the values of the surface integrals $\underset{6}{c}_m$. To do this we must first insert in (15.2) the values previously found for $\underset{4}{\gamma}_{00}$, $\underset{4}{\gamma}_{mn}$ and $\underset{5}{\gamma}_{0n}$ and then it is a matter of calculating the contributions of the resulting terms one by one and adding the expressions obtained. The general technique is similar to that used in 11 for the evaluation of $\underset{4}{c}_m$ but considerably more complicated.

On making use of our right to take $\underset{4}{c}_m$ to be zero, we may express the result in the form

$$
\begin{aligned}
\underset{6}{\overset{1}{c}}_m &= \frac{1}{4\pi}\int^1 2\underset{6}{\Lambda}_{mn}\cos(\mathbf{n}\cdot\mathbf{N})\,dS \\
&= -4\overset{1}{m}\overset{2}{m}\Bigg\{\Bigg[\dot\eta^s\dot\eta^s + \tfrac{3}{2}\dot\zeta^s\dot\zeta^s - 4\dot\eta^s\dot\zeta^s - 4\frac{\overset{2}{m}}{r} - 5\frac{\overset{1}{m}}{r}\Bigg]\frac{\partial}{\partial\eta^m}\left(\frac{1}{r}\right) \\
&\qquad [4\dot\eta^s(\dot\zeta^m - \dot\eta^m) + 3\dot\eta^m\dot\zeta^s - 4\dot\zeta^s\dot\zeta^m]\frac{\partial}{\partial\eta^s}\left(\frac{1}{r}\right) + \frac{1}{2}\,\frac{\partial^3 r}{\partial\eta^s\,\partial\eta^r\,\partial\eta^m}\dot\zeta^s\dot\zeta^r\Bigg\}.
\end{aligned}
\tag{16.1}
$$

17. The Main Deviation from the Newtonian Equations of Motion. In order to obtain the equations of motion belonging to this stage of our approximation, we must write

$$\lambda^4 \underset{4}{\overset{k}{C}}_m + \lambda^6 \underset{6}{\overset{k}{C}}_m = 0 \qquad k = 1, 2 \tag{17.1}$$

and then must reabsorb the λ's by going over to the old time x^0 instead of the auxiliary time $\tau = \lambda x^0$ and by introducing a corresponding change in mass from m to M, where $M = \lambda^2 m$. There will be no confusion if we keep the old notation for the new quantities so that now $\dot{\xi} = d\xi/dx^0$ instead of $d\xi/d\tau$, and m is written for the new mass M. And with this convention we may write the equations of motion (17.1), by means of (11.10) and (16.1), in the form

$$\begin{aligned} \ddot{\eta}^m - \overset{2}{m}\frac{\partial(1/r)}{\partial\eta^m} = \overset{2}{m}\Bigg\{&\left[\dot{\eta}^s\dot{\eta}^s + \tfrac{3}{2}\dot{\zeta}^s\dot{\zeta}^s - 4\dot{\eta}^s\dot{\zeta}^s - 4\frac{\overset{2}{m}}{r} - 5\frac{\overset{1}{m}}{r}\right]\frac{\partial}{\partial\eta^m}(1/r) \\ &[4\dot{\eta}^s(\dot{\zeta}^m - \dot{\eta}^m) + 3\dot{\eta}^m\dot{\zeta}^s - 4\dot{\zeta}^s\dot{\zeta}^m]\frac{\partial}{\partial\eta^s}(1/r) + \frac{1}{2}\frac{\partial^3 r}{\partial\eta^s\,\partial\eta^r\,\partial\eta^m}\dot{\zeta}^s\dot{\zeta}^r\Bigg\}. \end{aligned} \tag{17.2}$$

The equations of motion for the other particle are obtained by replacing $\overset{1}{m}$, $\overset{2}{m}$, η, ζ by $\overset{2}{m}$, $\overset{1}{m}$, ζ, η.

These equations, giving the relativistic motion of two massive gravitating bodies, constitute the main result of our calculations from the point of view of practical application.

These equations have since been integrated by H. P. Robertson, whose results are given in the following note on "The Two Body Problem in General Relativity," Math. Ann. 39, p. 101 (1938).

We should like to thank Professor Robertson for the very kind interest he took in this problem and for his help.

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