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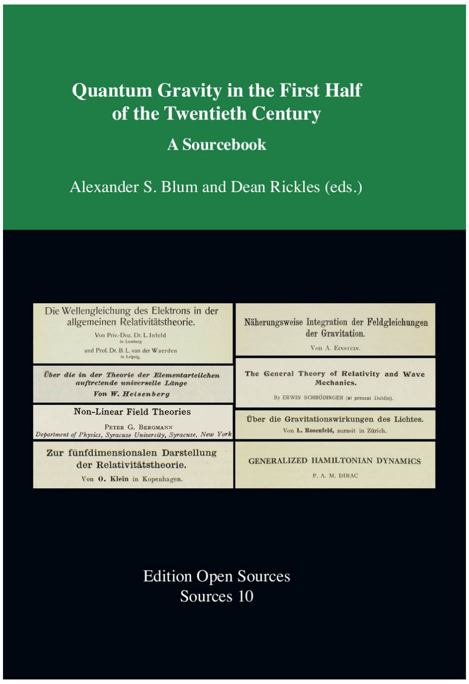

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Chapter 16
Erwin Schrödinger (1940): The General Theory of Relativity and Wave Mechanics

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The General Theory of Relativity and Wave Mechanics.

By ERWIN SCHRÖDINGER (at present Dublin).

1. *Curved space-time — the model of matter.* — The most important discoveries are those which in the course of time tend to become tautological. The logical content of Newton's first two laws of motion was to state, that a body moves uniformly in a straight line, unless it does something else and that in the latter case we agree upon *calling force* its acceleration multiplied by an individual constant. The great achievement was, to concentrate attention on the *second* derivatives — to suggest that *they* — not the first or third or fourth, not any other property of the motion — ought to be accounted for by the environment. It is very often more essential just to point out the kind of conception that matters and to emphasize it by claiming for it a name of its own, than to find out in detail the laws which control it.

The fundamental statements of Einstein's theory of gravitation are of a similar kind. The equations

$$G_{ik} - \tfrac{1}{2} g_{ik} G = -8\pi T_{ik} \qquad (1)$$

state, that the contracted curvature tensor G_{ik} is either zero or not (1) and that, when and where it is not, we *call matter* (T_{ik}) the left hand side of equations (1). Thus the equations amount to a definition of *matter* (T_{ik}), on which they impose the *conservation laws*, in virtue of a well known identity, satisfied by the G_{ik} of *any* metric g_{ik}. *They do not impose any restriction whatsoever on the metric* (g_{ik}). Yet this definition is all-important. Matter is no longer the content of the receptacle space-time, is no longer the actor who performs in time on the stage of space. Matter and space-time have really merged into one. It is sometimes said, that matter determines the curvature of space-time. But the most advisable attitude is, I think, to reserve the expression of *matter* to indicate the object of our direct observation and to regard *curved space-time* as the picture or *model* we form of this object in our minds, bearing thus the same relation to matter as the Rutherford-Bohr model to the atom of observation. Equation (1) enounces the choice of the model. Our observations on matter determine *us* to equip the model we have chosen with such features (curvature) as make it fit in with observation.

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(1) See A. S. Eddington, The mathematical theory of relativity, Ch. IV, sect. 54.

2. *The model gives all forces the same standing.* — Though I do not expect to be opposed in what I just said, I wish to point out that it compels us to change the wording of some familiar statements. It is very often said that on the atomic scale and even on the macroscopic scale no influence of gravitation is perceptible, apart from the intense gravitational fields of the huge celestial bodies. Now look at that book lying on your desk. Why does it not fall in spite of the gravitational pull of the planet? On account of the elastic pressure, equal and of opposite sign, which the table exerts on it. Of what kind is that pressure? You may begin to tell me of deformed cristal-lattices, deformed electronic states, particle-encounters, exchange-forces, etc. To which I reply, that no doubt you are right. But whether your details are correct or not, the result of all that is certainly, that some T_{ik}, very well known from the theory of elasticity, are aroused in the table and in the book. These T_{ik} form part of the right hand side of equation (1) and compel us to equip our space-time continuum whith the corresponding curvature G_{ik}. And that is essential for the book being maintained in its place in our model. The circumstances which would cause the book to fall, if the table were not there, and those which actually prevent it from falling are thus of exactly the same kind, they have exactly the same standing. If it is said, as it often is, that the essential thing about Einstein's theory is to explain gravitation by the curvature of space-time, then one has to declare, that all forces — i. e. all that was formerly called force — are gravitational. Gravitation in this sense is not to be coordinated with forces « of other origin », it is the comprehensive conception of them all.

Take an electron, moving in an electrostatic field in, otherwise, empty space. Why does it not follow the geodesic? The usual answer is: because it is charged and there is the electric pull on it. The correct answer is: 1) Only in empty space ($T_{ik} = 0$) is a particle, under certain assumptions, expected to follow the geodesic. Our electron is moving through matter ($T_{ik} \neq 0$). 2) It is true, that other (sc. neutral) particles would under the same circumstances follow the geodesic; that our electron does not, is due to certain modifications (additional T_{ik}) which it produces by reacting with the matter through which it moves. The case resembles that of a gun-ball moving in air, which, for similar reasons, does not follow the parabola.

Whereas the gravitational field which in the popular sense of the word is produced by the energy or mass (T_{44}) of an electromagnetic field is practically negligible, we see from the preceeding « example », that the curvature corresponding to components like T_{11}, T_{23} etc. is appreciable and rather efficient. That is partly due to the fact, that in the case of an electromagnetic field these components are of the same order of magnitude as T_{44} (which in ordinary matter they are usually not); partly to the particularly favourable spatial distribution of those T_{ik} which are aroused by the interaction of the two electric fields, that of the electron and that through which it moves.

3. *The model does not determine particle motion.* — It was emphasized in § 1, that the equations (1) do not impose any restriction on the metric (g_{ik}). Let us choose an arbitrary metric (g_{ik}) and calculate its matter (T_{ik}) from (1). Let us assume, that there are empty regions, i. e. with $T_{ik} = 0$. Now choose, quite arbitrarily, a four-dimensional track, which passes also through empty regions. Choose among the *neighbouring* metrics $g_{ik} + \delta g_{ik}$ one for which the $T_{ik} + \delta T_{ik}$, calculated from (1), differ from the T_{ik} only in the vicinity of the chosen track. That is certainly possible, for you might restrict the δg_{ik} themselves to the vicinity of the track (though that would be a very special choice).

In those parts where the track passes through regions that were empty in the metric g_{ik} it will, in the metric $g_{ik} + \delta g_{ik}$, have certain features of the world line of a moving particle. But since the track was arbitrarily chosen, it need not, of course, be a geodesic of the original metric. It may include space-like directions, may (by passing through such) change from positive to negative time-like directions, may begin or end suddenly in previously empty space-time, may exhibit any desired kind of bifurcations etc.

That the field equations (1) have no tendency whatsoever of guiding a small disturbance in vacuo along a geodesic, is a well known fact(1). (I have insisted on recalling it here, because it is liable to be veiled again and again by the treatment in some text books.) What then do they tell us about this case ? Nothing beyond the conservation identities. For an isolated disturbance in vacuo that means *the parallel displacement of the resultant energy momentum-vector along the path* — provided that there is a path, i. e. as long as the disturbance keeps together.

Neither do the field equations take charge of keeping the disturbance together — they could not, for they have to be capable of describing events, where it is not the case. Nor do they say anything about the track, because any four-vector can be parallelly displaced along any track. One thing can be said : if you choose to let the path begin or end in vacuo, you may be sure to find the energy-momentum vector zero at every cross section. So this case would necessarily introduce

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(1) A. S. Eddington, The mathematical Theory of Relativity, Ch, 4, sect. 56.

negative mass density. A bifurcation of such a track of total mass zero into a positive and a negative branch would be perfectly admitted, and so would the confluence of two such tracks be, e. g. with subsequent annihilation.

In order to preclude these and other odd possibilities, contrary to observation, current theory introduces the following assumptions : 1) mass is always positive, 2) it is concentrated in particles that are ordinarily indivisible, 3) the (four dimensional) tangent of the track continues to point in the direction of the resultant energy-momentum vector. — The third assumption, when combined with the conservation law, introduces the *geodesic* track rather openly as a distinct hypothesis. (It may be pointed out, though, that it would be pretty difficult to introduce, in an equally simple way, *another* law for the track.)

4. *There is room for wave-mechanics to supplement the model.* — These points have, of course, always formed an integrant part of the theory, so much so, that one is liable to take them for granted and to overlook the fact, that the field equations need any supplement at all. Indeed the latter was an almost trivial and straightforward loan from the particle-picture of matter, which was well developed at that time, whereas the discovery of the field equations was the achievement of unusual geniality and painstaking.

There can be little doubt, that from the outlook we have reached today we have to convert this loan by putting wave-mechanics in the place of particle mechanics. What I wish to emphasize, is *that there is room and even need for supplementing the general theory of relativity by wave-mechanics*. You could not expect relativity theory to produce out of its own something of that kind, say e.g. in the way that gravitational waves turned out from equations (1) to follow laws similar to those of material waves. If you wish them to do that, you have to introduce it as an explicit assumption, and you are allowed to do it. For the field equations themselves have just as little control of the behaviour of a wave-like disturbance as of that of a particle-like disturbance.

Wave-mechanics seems to be very fit for furnishing the wanted. Anyone of the current wave-equations, when placed in an arbitrary Riemannian space and disencumbered of the terms corresponding to explicit forces, will — in virtue of the Fermat-Hamilton-principle — describe a wave propagation along the geodesics of that space. Moreover in the case of parcel-like solutions, which are always available, the theory affords a definition of the momentum vector, which actually undergoes, from the very laws of the wave motion, a parallel displacement *in its own direction*. It is true, that things are not quite as simple as I have made them look by these statements — inter alia, because they primarily refer to threedimensional space. But a generalisation to include time is quite possible.

One has the impression, that wave-mechanics meets the requirements of general relativity even in a more congenial way than particle mechanics did. That a particle-like solution does not, as a rule, keep together for ever and that, moreover, solutions of entirely different type exist, need not worry us, because we know what it means (principle of indeterminacy). Further we know, that the demarcation of individuals, corresponding to the observed discontinuity of matter, finds an entirely different expression in wave-mechanics, not by delimiting the individuals in space, but by the discreteness in energy-momentum space (line-spectrum).

5. *Two geometrical aspects of matter : the differential and the integral one.* — From our present outlook on Nature there is, I think, no other way of accounting for the atomicity of matter than by admitting the eigenvalues of the wave-equation to be discrete, to form a line-spectrum. And there is, I believe, no other general means of having them so, than by assuming space to be closed and finite. If these conclusions are accepted, they open far reaching prospects.

The proper modes of a given closed space, since they do not depend on boundary conditions, may be regarded as a purely geometrical property of that space. The wave aspect connects matter with these proper modes, thus with pure geometry — as the space-time model of general relativity also does. But the two connexions are of entirely different kind. The one is a local and differential one : matter here and now is described by the curvature here and now. The other is an integral connexion. The very nature of the matter which forms the Universe is determined by the shape and size of the Universe as a whole. I believe that these two aspects are not contradictory but complementary.

There is an interesting coincidence to confirm these views. All observed masses are positive. Positive mass has, to say the least, a marked tendency of producing a positive curvature of *space* — such is, at any rate, the current interpretation of the state of affairs within our galaxy. A continuum that in all its parts has on the average a positive curvature, cannot, I think, avoid singularities without being closed. But from the uncertainty principle we may conclude, that a closed configuration space, since it sets an upper limit to the uncertainty of position, entails a positive kinetic zero-point energy, which gives rise to positive mass. Thus closed space is, in a way, self-consistent. There is a remarkable general agreement between the two geometrical connexions in that they both suggest a positive density of matter — as is also observed.

6. *Attempts to account for the elementary mass.* — The ideas I have just expounded are due to A. S. Eddington (1). We ought to discriminate carefully between them and Eddington's elaborate attempt to carry them out. Let me in the simplest and most primitive terms (for which I alone am responsible) outline the difficulties met with.

A first naive trial would run thus. If R is the radius of a spherical Universe, configuration space is restricted to the linear dimension R. The momentum p of the lowest state will therefore be of the order of magnitude

$$p = \frac{h}{2\pi R} \cdot \qquad (2)$$

We associate with p an energy $\frac{p^2}{2\,m}$ and put it equal to the rest-or zero-point-energy $m\,c^2$. Thus

$$\frac{1}{2m}\left(\frac{h}{2\pi R}\right)^2 = mc^2$$

or

$$m = \frac{h}{2\pi Rc\sqrt{2}} \cdot \qquad (3)$$

To give the mass of the proton, the radius of the Universe, R, would have to be of the order $h/mc \approx 10^{-13}$ cm, wich is pure nonsense.

We next introduce, following Eddington, the idea, that the universe is a system of N particles, observing Pauli's exclusion principle. We wish to put the energy of the lowest state *of this N-particle system* equal to the *total* rest-energy Nmc^2. It amounts to the same as to order of magnitude (which alone interests us for the moment), if we put mc^2 equal to the energy of the N^{th} (single particle) level. A well known formula

$$N = \frac{4\pi}{3}\frac{V}{\lambda^3} \qquad (4)$$

connects the wave length λ of the N^{th} level with N and with the volume $V = 2\,\pi^2 R^3$. From λ follows p by de Broglie's relation : $p = \frac{h}{\lambda} \cdot$

(1) « Protons and Electrons », Cambridge 1936.

So we have

$$N = \frac{4\pi}{3} \frac{2\pi^2 R^3}{h^3} p^3,$$

and therefore

$$\frac{1}{2m} p^2 = \frac{1}{2m} \frac{h^2 (3N)^{2/3}}{4\pi^2 R^2} = mc^2$$

or

$$m = \frac{h(3N)^{1/3}}{2\pi Rc\sqrt{2}}. \tag{5}$$

The improvement is, that we get no longer R itself, but

$$\frac{R}{N^{1/3}} \approx \frac{h}{mc} \approx 10^{-13}\,\text{cm}, \tag{6}$$

that is to say, *the mean interstice* between the particles would have to be of the order of magnitude known to prevail *in the nucleus*. Though this is far from the actual state of the universe (where the average density corresponds to one or two protons per liter), it is rather striking, that we find a great part of the world's material (maybe the bulk of it) in « lumps » of this completely degenerated state. Of course, why the observed mass of a « free » proton, e. g. that of a hydrogen nucleus in CH_4, when investigated in the mass spectrograph, should be essentially determined by the degenerated state in the heavier nuclei, would appear inscrutable.

I should like to mention, that I have tried to arrive at something like formula (5) in many more elaborate and more sophisticated ways than described above, in the faint hope of letting *the square root* of N slip in, instead of the cube root. I have really tried hard. But apart from inappreciable numerical factors I always get the same result.

Eddington arrives at the square root, which makes the formula agree with probable values of R ($\approx 10^{27}$ cm) and N ($\approx 10^{79}$). His main and elaborate derivation, which *accepts* the exclusion principle, is beyond my understanding and I beg to be excused for skipping it. A simple and clear-cut one *foregoes* the exclusion principle, accepts (2), but declares, that the observed rest energy mc^2 corresponds not to the observed particle's zero point energy, but rather to that of the centroid of all the other particles together. (The idea is, I think, that the other particles form the material coordinate frame.) If this is accepted, the desired result follows at once, because, with haphazard directions,

$$p\sqrt{N} = \frac{h}{2\pi R}\sqrt{N}$$

is the momentum of the centroid and now you put, as before,

$$\frac{1}{2m}\left(\frac{h}{2\pi R}\sqrt{N}\right)^2 = mc^2,$$

thus

$$m = \frac{h\sqrt{N}}{2\pi Rc\sqrt{2}}.$$

7. *Conclusion.* — These attempts are hardly encouraging. But we must bare in mind the unusual difficulty of the task. In arguing about the Universe, with a view to explain how the laws we know from experiments on a small scale come about, we have nothing to argue with than those small-scale-laws themselves. So we are in the sort of position of that worthy baronet who (said he) dragged himself out of the swamp by pulling his own pigtail.

There is a small length h/mc ($\approx 10^{-13}$ cm), intimately connected with the elementary unit of mass m (that of the proton or neutron) and characteristic of the structure of aggregations of elementary masses (= structure of the nucleus). There is, as a counterpart, a large length $2\pi R$ or R ($\approx 10^{27}$ cm), characteristic of the structure of the Universe. Between the two there is a pretty well established semi-empirical relation, with the number N ($\approx 10^{79}$) of elementary masses that form the Universe intervening, viz.

$$\frac{h}{mc} \approx \frac{R}{\sqrt{N}}. \qquad (7)$$

For all that I know, the line of thought reported in the earlier sections of this paper, is the only one to suggest any connection at all between the structure of the Universe as a whole and the structure of the matter that constitutes it. So we have hardly any choice than to follow it, unless we acquiesce in letting sleeping dogs lie.