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Chapter 34
P. A. M. Dirac (1950): Generalized Hamiltonian Dynamics

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GENERALIZED HAMILTONIAN DYNAMICS

P. A. M. DIRAC

1. Introduction. The equations of dynamics were put into a general form by Lagrange, who expressed them in terms of a set of generalized coordinates and velocities. An alternative general form was later given by Hamilton, in terms of coordinates and momenta. Let us consider the relative merits of the two forms.

With the Lagrangian form the requirements of special relativity can very easily be satisfied, simply by taking the action, i.e. the time integral of the Lagrangian, to be Lorentz invariant. There is no such simple way of making the Hamiltonian form relativistic.

For the purpose of setting up a quantum theory one must work from the Hamiltonian form. There are well-established rules for passing from Hamilton's dynamics to quantum dynamics, by making the coordinates and momenta into linear operators. The rules lead to definite results in simple cases and, although they cannot be applied to complicated examples without ambiguity, they have proved to be adequate for practical purposes.

Thus both forms have their special values at the present time and one must work with both. The two forms are closely connected. Starting with any Lagrangian one can introduce the momenta and, in the case when the momenta are independent functions of the velocities, one can obtain the Hamiltonian. The present paper is concerned with setting up a more general theory which can be applied also when the momenta are not independent functions of the velocities. A more general form of Hamiltonian dynamics is obtained, which can still be used for the purpose of quantization, and which turns out to be specially well suited for a relativistic description of dynamical processes.

2. Strong and weak equations. We consider a dynamical system of N degrees of freedom, described in terms of generalized coordinates $q_n (n = 1,2, \ldots, \text{N})$ and velocities dq_n/dt or $\dot{q}_n$. We assume a Lagrangian L, which for the present can be any function of the coordinates and velocities

$$L \equiv L(q, \dot{q}). \tag{1}$$

We define the momenta by

$$p_n = \partial L/\partial \dot{q}_n. \tag{2}$$

For the development of the theory we introduce a variation procedure, varying each of the quantities q_n, $\dot{q}_n$, p_n independently by a small quantity δq_n, $\delta \dot{q}_n$, δp_n of order ϵ and working to the accuracy of ϵ. As a result of this

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variation procedure equation (2) will get violated, as its left-hand side will be made to differ from its right-hand side by a quantity of order ϵ. We shall now have to distinguish between two kinds of equations, equations such as (2) which get violated by a quantity of order ϵ when we apply the variation, and equations which remain valid to the accuracy ϵ under the variation. Equation (1) will be of the latter kind, since the variation in L will equal, by definition, the variation of the function $L(q,\dot{q})$. The former kind of equation we shall call a *weak equation* and write with the usual equality sign $=$, the latter we shall call a *strong equation* and write with the sign $\equiv$.

We have the following rules governing algebraic work with weak and strong equations:

$$\text{if} \quad A \equiv 0 \quad \text{then} \quad \delta A = 0;$$
$$\text{if} \quad X = 0 \quad \text{then} \quad \delta X \neq 0;$$

in general. From the weak equation $X = 0$ we can deduce

$$\delta X^2 = 2X\delta X = 0,$$

so we can deduce the strong equation

$$X^2 \equiv 0.$$

Similarly, from two weak equations $X_1 = 0$ and $X_2 = 0$ we can deduce the strong equation

$$X_1 X_2 \equiv 0.$$

It may be that the N quantities $\partial L/\partial \dot{q}_n$ on the right-hand side of (2) are all independent functions of the N velocities $\dot{q}_n$. In this case equations (2) determine each $\dot{q}$ as a function of the q's and p's. This case will be referred to as the *standard case*, and is the only one usually considered in dynamical theory.

If the $\partial L/\partial \dot{q}$'s are not independent functions of the velocities, we can eliminate the $\dot{q}$'s from equations (2) and obtain one or more equations

$$\phi(q, p) = 0 \tag{3}$$

involving only q's and p's. We may suppose equation (3) to be written in such a way that the variation procedure changes ϕ by a quantity of order ϵ, since if it changes ϕ by a quantity of order ϵ^k, we have only to replace ϕ by $\phi^{1/k}$ in (3) and the desired condition will be fulfilled. We now have equation (3) violated by the order ϵ when we apply the variation, so it is correctly written as a weak equation.

We shall need to use a complete set of independent equations of the type (3), say

$$\phi_m(q, p) = 0, \qquad m = 1,2, \ldots, \text{M}. \tag{4}$$

The condition of independence means that none of the ϕ's is expressible linearly in terms of the others, with functions of the q's and p's as coefficients. The condition of completeness means that any function of the q's and p's which vanishes on account of equations (2) and changes by the order ϵ with the variation procedure is expressible as a linear function of the ϕ_m with functions of the q's and p's as coefficients.

We may picture the relationship of strong and weak equations in the following way. Take the $3N$ dimensional space with the q's, $\dot{q}$'s and p's as coordinates. In this space there will be a $2N$ dimensional region in which equations (2) are satisfied. Call it the region R. Equations (4) will also be satisfied in this region, as they are consequences of (2). Now consider all points of the $3N$ dimensional space which are within a distance of order ϵ from R. They will form a $3N$ dimensional region like a shell with a thickness of order ϵ. Call this the region R_ϵ. *A weak equation holds in the region R, a strong equation holds in the region R_ϵ.*

3. The Hamiltonian. The Hamiltonian H is defined by

$$H \equiv p_n \dot{q}_n - L, \tag{5}$$

where a summation is understood over all values for a repeated suffix in a term. We have

$$\begin{aligned} \delta H &= \delta(p_n \dot{q}_n - L) \\ &= p_n \delta \dot{q}_n + \dot{q}_n \delta p_n - \partial L/\partial q_n . \delta q_n - \partial L/\partial \dot{q}_n . \delta \dot{q}_n \\ &= \dot{q}_n \delta p_n - \partial L/\partial q_n . \delta q_n . \end{aligned} \tag{6}$$

We find that δH does not depend on the $\delta \dot{q}$'s. This important result holds whether we have the standard case or not.

Equation (5) gives a definition for H as a function of the q's, $\dot{q}$'s and p's, holding throughout the $3N$ dimensional space of q's, $\dot{q}$'s, and p's. We shall use the definition only in the region R_ϵ, and in this region the result (6) holds, to the first order. This means that, if we keep the q's and p's constant and make a first-order variation in the $\dot{q}$'s, the variation in H will be of the second order. Thus if we keep the q's and p's constant and make a finite variation in the $\dot{q}$'s, keeping all the time in the region R_ϵ (which is possible when we do not have the standard case), the variation in H will be of the first order. If we keep in the region R, the variation in H will be zero. It follows that in the region R, H is a function of the q's and p's only. Calling this function $\mathfrak{H}(q, p)$, we have the weak equation

$$H = \mathfrak{H}(q, p) \tag{7}$$

holding in the region R. In the standard case the function $\mathfrak{H}$ is the ordinary Hamiltonian.

Starting from a point in R and making a general variation, we have from (6)

$$\delta(H - \mathfrak{H}) = \left(\dot{q}_n - \frac{\partial \mathfrak{H}}{\partial p_n}\right)\delta p_n - \left(\frac{\partial L}{\partial q_n} + \frac{\partial \mathfrak{H}}{\partial q_n}\right)\delta q_n .$$

Thus $\delta(H - \mathfrak{H})$ depends only on the δq's and δp's. If the variation is such that we stay in the region R, then of course $\delta(H - \mathfrak{H}) = 0$. Thus $\delta(H - \mathfrak{H})$ vanishes for any variation of the q's and p's such that one can choose the $\delta \dot{q}$'s so as to preserve equations (2). The only restriction this imposes on the δq's and δp's is that they must preserve equations (4), i.e. they must lead to $\delta \phi_m = 0$ for all m. Thus $\delta(H - \mathfrak{H})$ is zero for any values δq, δp that make $\delta \phi_m = 0$, and hence for arbitrary δq, δp

$$\delta(H - \mathfrak{H}) = v_m \delta\phi_m \tag{8}$$

with suitable coefficients v_m. These coefficients will be functions of the q's, $\dot{q}$'s and p's, and with the help of (2) can be expressed as functions of the q's and $\dot{q}$'s only. We now get

$$\delta(H - \mathfrak{H} - v_m\phi_m) = \delta(H - \mathfrak{H}) - v_m\delta\phi_m - \phi_m\delta v_m = 0$$

from (8) and (4), and hence

$$H \equiv \mathfrak{H} + v_m\phi_m. \tag{9}$$

We have here a strong equation, holding to the first order in the region R_ϵ, in contradistinction to the weak equation (7), which holds only in R.

Equation (8) gives

$$\begin{aligned} \delta H &= \delta\mathfrak{H} + v_m\delta\phi_m \\ &= \frac{\partial\mathfrak{H}}{\partial p_n}\,\delta p_n + \frac{\partial\mathfrak{H}}{\partial q_n}\,\delta q_n + v_m\left(\frac{\partial\phi_m}{\partial p_n}\,\delta p_n + \frac{\partial\phi_m}{\partial q_n}\,\delta q_n\right). \end{aligned}$$

Comparing this with (6), we get

$$\dot{q}_n = \frac{\partial\mathfrak{H}}{\partial p_n} + v_m\,\frac{\partial\phi_m}{\partial p_n}\,, \tag{10}$$

$$-\frac{\partial L}{\partial q_n} = \frac{\partial\mathfrak{H}}{\partial q_n} + v_m\,\frac{\partial\phi_m}{\partial q_n}\,. \tag{11}$$

Equations (10) give the $\dot{q}$'s in terms of the q's, p's and v's. They show that the 2N variables $q_n, \dot{q}_n$ can be expressed in terms of the 2N + M variables q_n, p_n, v_m. Between these 2N + M variables there exist the M relations (4). There cannot be any other relations between these variables, as otherwise the 2N variables $q_n, \dot{q}_n$ would not be independent. Thus the v's must each be independent of the q's, p's and other v's. The v's can be considered as a kind of velocity variables, which serve to fix those $\dot{q}$'s that cannot be expressed in terms of q's and p's.

When we work with the Hamiltonian form of dynamics we use as basic variables the q's, p's and v's, between which certain relations (4) are assumed to exist, and which are otherwise independent. These variables will be called the *Hamiltonian variables*.

4. The equations of motion. We assume the usual Lagrangian equations of motion as weak equations,

$$\dot{p}_n = \partial L/\partial q_n. \tag{12}$$

By substituting for the p's in (12) their values given by (2), we get equations involving the accelerations $\ddot{q}_n$. In the standard case these equations will determine all the $\ddot{q}$'s in terms of the q's and $\dot{q}$'s. In the case with M equations (4), the equations of motion will give us only N − M equations for the $\ddot{q}$'s. The remaining M equations of motion will tell us how the ϕ_m's vary with time.

For consistency the ϕ_m's must remain zero. These consistency conditions will be examined later.

With the help of (11) the equations of motion (12) take the form

$$\dot{p}_n = -\frac{\partial \mathfrak{H}}{\partial q_n} - v_m \frac{\partial \phi_m}{\partial q_n}. \tag{13}$$

Equations (13) together with (10) constitute the Hamiltonian equations of motion. They are fixed by the function $\mathfrak{H}$ and the equations $\phi_m = 0$. The Hamiltonian equations of motion give us the $\dot{q}$'s and $\dot{p}$'s in terms of the Hamiltonian variables q, p, v. They give us no direct information about the $\dot{v}$'s, but will give us some information indirectly when we examine the consistency conditions.

The Hamiltonian equations of motion can be expressed more easily with the help of the Poisson bracket notation. Any two functions ξ and η of the q's and p's have a P. b. $[\xi, \eta]$, defined by

$$[\xi, \eta] \equiv \frac{\partial \xi}{\partial q_n}\frac{\partial \eta}{\partial p_n} - \frac{\partial \xi}{\partial p_n}\frac{\partial \eta}{\partial q_n}. \tag{14}$$

It is easily verified that the P. b. remains invariant under a transformation to new q's and p's, in which the new q's are any independent functions of the original q's and the new p's are defined by the new equations (2) with L expressed in terms of the new q's and their time derivatives. This invariance property gives the P. b. its importance.

P. b.'s are subject to the following laws, which are easily verified from the definition:

$$\begin{cases} [\xi, \eta] \equiv -[\eta, \xi], \\ [\xi, f(\eta_1, \eta_2, \ldots)] \equiv \dfrac{\partial f}{\partial \eta_1}[\xi, \eta_1] + \dfrac{\partial f}{\partial \eta_2}[\xi, \eta_2] + \ldots, \\ [\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] \equiv 0. \end{cases} \tag{15}$$

In the second of these laws f is any function of various quantities $\eta_1, \eta_2, \ldots$, each of which is a function of the q's and p's. The third law, known as Poisson's identity, applies to any three functions ξ, η, ζ of the q's and p's.

It is desirable to extend the notion of P. b.'s to include functions of the $\dot{q}$'s which are not expressible in terms of the q's and p's. We assume these more general P. b.'s are subject to the laws (15) but are otherwise arbitrary. Alternatively, we may assume that the $\dot{q}$'s are arbitrary functions of the q's and p's, and the laws (15) can then be deduced with ξ, the η's and ζ involving the $\dot{q}$'s.

From a strong equation $A \equiv 0$ we can infer the weak equations

$$\frac{\partial A}{\partial q_n} = 0, \quad \frac{\partial A}{\partial \dot{q}_n} = 0, \quad \frac{\partial A}{\partial p_n} = 0,$$

and hence, by an application of the second of the laws (15),

$$[\xi, A] = 0$$

for any ξ. We may have $[\xi, A] \equiv 0$, (for example when $A \equiv 0$ by definition) but this is not necessarily so. From a weak equation $X = 0$ we cannot infer $[\xi, X] = 0$ in general.

If g is any function of the q's and p's, we have from (10) and (13)

$$\dot{g} = \frac{\partial g}{\partial q_n}\left(\frac{\partial \mathfrak{H}}{\partial p_n} + v_m \frac{\partial \phi_m}{\partial p_n}\right) - \frac{\partial g}{\partial p_n}\left(\frac{\partial \mathfrak{H}}{\partial q_n} + v_m \frac{\partial \phi_m}{\partial q_n}\right)$$

$$= [g, \mathfrak{H}] + v_m [g, \phi_m]. \tag{16}$$

This is the general Hamiltonian equation of motion. It may also be written, with the help of (4), as

$$\dot{g} = [g, \mathfrak{H}] + v_m [g, \phi_m] + [g, v_m]\, \phi_m = [g, H], \tag{17}$$

when it is the same as the usual Hamiltonian equation of motion in P. b. notation.

5. Homogeneous velocities. The theory takes a specially simple form in the case when the Lagrangian is homogeneous of the first degree in the velocities. The momenta defined by (2) are then homogeneous of degree zero in the $\dot{q}$'s and so depend only on the ratios of the $\dot{q}$'s. Since there are N p's and only N $-$ 1 independent ratios of the $\dot{q}$'s, the p's now cannot be independent functions of the $\dot{q}$'s and there must be at least one relation (4) connecting the q's and p's. The case when there is only one relation between the q's and p's may now be considered as the standard case.

From Euler's theorem we have

$$L \equiv \dot{q}_n\, \partial L/\partial \dot{q}_n \tag{18}$$

and hence

$$L = \dot{q}_n p_n$$

so that

$$H = 0. \tag{19}$$

This weak equation holding in the region R allows us to take $\mathfrak{H} \equiv 0$, so that (9) becomes

$$H \equiv v_m \phi_m. \tag{20}$$

The general equation of motion (16) is now

$$\dot{g} = v_m [g, \phi_m]. \tag{21}$$

The Hamiltonian equations of motion are now fixed entirely by the equations $\phi_m = 0$.

Equation (21) is homogeneous in the v's on the right-hand side. Given any solution of the equations of motion, one can obtain another solution from it by multiplying all the v's by a factor γ, which may vary arbitrarily with the time. The new solution will have the time rate of change of all dynamical variables multiplied by the factor γ. The new solution would be obtained from the previous solution if we replaced the time t by a new independent variable τ such that $dt/d\tau = \gamma$. The new independent variable is completely arbitrary: it can be any function of t and the q's and $\dot{q}$'s. Thus, given any solution of the equations of motion, we can get another solution from it by replacing t by an arbitrary τ, so *the equations of motion give us no information about the independent variable.* This is an important feature of dynamical theory with homogeneous velocities, and makes it specially convenient for a relativistic treatment.

The Lagrangian for any dynamical system can be made to satisfy the condition for homogeneous velocities by taking the time t to be an extra coordinate q_0 and using the equation $\dot{q}_0 = 1$ to make the Lagrangian homogeneous of the first degree in all the velocities, including $\dot{q}_0$. The new Lagrangian equations of motion for all the q's can then be deduced, as has been shown by the author [1]. In this way we can get a new formulation for a general dynamical system in terms of homogeneous velocities. The new formulation gives all the equations of the old formulation except the equation $\dot{q}_0 = 1$. If we want to have this equation in the new formulation we may assume it as a supplementary condition, not derivable from the equations of motion but consistent with them. We can, however, very well dispense with it, as its only effect is to fix the independent variable, which would otherwise be arbitrary in the homogeneous velocity formulation.

Thus we may confine ourselves to the homogeneous velocity theory without losing any generality. We shall do this in future as it leads to somewhat simpler equations, and use the dot to denote differentiation with respect to an arbitrary independent variable τ.

6. The consistency conditions. For consistency the equations of motion must make each of the ϕ_m remain zero. Thus, putting $\phi_{m'}$ for g in (21), we get

$$v_m [\phi_m, \phi_{m'}] = 0. \tag{22}$$

Let us suppose the equations (22) to be reduced as far as possible with the help of the set of equations (4). The reduction may involve the cancellation of factors when we can assume these factors do not vanish. The resulting equations must each be of one of four types.

Type 1. It involves some of the variables v.

Type 2. It is independent of the v's but involves some of the variables p and q. It is thus of the form

$$\chi(q, p) = 0 \tag{23}$$

and is independent of the equations (4).

Type 3. It reduces to $0 = 0$.

Type 4. It reduces to $1 = 0$.

An equation of type 2 leads to a further consistency condition, since we must have χ remaining zero. Putting χ for g in (21), we get

$$v_m[\phi_m, \chi] = 0. \tag{24}$$

This equation, reduced as far as possible with the help of equations (4) and any equations (23) that we already have, will again be of one of the four types.

If it is of type 2 it will lead to yet another consistency condition. We continue in this way with each equation of type 2 until it leads to an equation of another type.

If any of the equations obtained in this way is of type 4, the equations of motion are inconsistent. This case is of no interest and will be excluded in future. Equations of type 3 are automatically satisfied. We are left with the equations of types 1 and 2 to fit into the theory.

Let us call the complete set of equations of type 2

$$\chi_k(q,p) = 0, \qquad k = 1, 2, \ldots, \text{K}. \tag{25}$$

We may suppose the functions χ_k to be chosen, like the ϕ_m in (4), so that their variations are of order ϵ. Equations (25) are then correctly written as weak equations. These further weak equations will reduce the region R, in which all weak equations hold, so as to have only $2\text{N} - \text{K}$ dimensions. The region R_ϵ will also get reduced, as it will now consist of all points within a distance of order ϵ from the new region R.

For studying the equations of type 1, it is convenient to introduce some new concepts. We define one of the quantities ϕ_m to be a *first class* ϕ if its P. b. with every ϕ and χ vanishes. Thus $\phi_{m'}$ is first class if

$$\begin{aligned} [\phi_{m'}, \phi_m] &= 0, \qquad m = 1, 2, \ldots, \text{M}, \\ [\phi_{m'}, \chi_k] &= 0, \qquad k = 1, 2, \ldots, \text{K}. \end{aligned} \tag{26}$$

These equations need hold only in the weak sense, which means that they need hold only as consequences of the equations $\phi_m = 0$, $\chi_k = 0$. Thus the left-hand sides of (26) must each equal in the strong sense some linear function of the ϕ_m and χ_k. A ϕ which does not satisfy all these conditions we call a *second class* ϕ.

We can make a linear transformation of the ϕ's of the form

$$\phi^*_m = \gamma_{mm'}\phi_{m'}, \tag{27}$$

where the γ's are any functions of the q's and p's such that their determinant does not vanish in the weak sense. The ϕ^*'s are then equivalent to the ϕ's for all the purposes of the theory.

Let us make a transformation of this kind so as to bring as many ϕ's as possible into the first class. Let us call the first class ϕ's that we then have ϕ_α's and the second class ones ϕ_β's, with $\beta = 1, 2, \ldots \text{B}$ and $\alpha = \text{B} + 1, \text{B} + 2, \ldots \text{M}$.

If $\phi_{m'}$ is first class, equation (22) is automatically satisfied. Further, in equations (22) and (24) we can restrict ϕ_m to be second class, as first class ϕ_m's contribute zero. Thus the surviving equations (22) and (24) will read

$$\begin{aligned} v_\beta[\phi_\beta, \phi_{\beta'}] &= 0, \qquad \beta, \beta' = 1, 2, \ldots, \text{B}, \\ v_\beta[\phi_\beta, \chi_k] &= 0, \qquad k = 1, 2, \ldots, \text{K}. \end{aligned} \tag{28}$$

These are all the equations of type 1. They show that *either* all the v_β's vanish *or* the matrix

$$(29)\qquad \left\| \begin{matrix} 0 & [\phi_1, \phi_2] & [\phi_1, \phi_3] & \dots & [\phi_1, \phi_B] & [\phi_1, \chi_1] & \dots & [\phi_1, \chi_K] \\ [\phi_2,\phi_1] & 0 & [\phi_2, \phi_3] & \dots & [\phi_2,\phi_B] & [\phi_2, \chi_1] & \dots & [\phi_2, \chi_K] \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \dots & \cdot \\ [\phi_B,\phi_1] & [\phi_B, \phi_2] & [\phi_B,\phi_3] & \dots & 0 & [\phi_B,\chi_1] & \dots & [\phi_B, \chi_K] \end{matrix} \right\|$$

is of rank less than B, in the weak sense.

It will now be proved that the first alternative is the correct one. Assume the matrix (29) is of rank U < B. Form the determinant

$$(30)\qquad D \equiv \begin{vmatrix} \phi_1 & 0 & [\phi_1,\phi_2] & [\phi_1,\phi_3] & \dots & [\phi_1,\phi_U] \\ \phi_2 & [\phi_2,\phi_1] & 0 & [\phi_2,\phi_3] & \dots & [\phi_2,\phi_U] \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ \phi_{U+1} & [\phi_{U+1},\phi_1] & [\phi_{U+1},\phi_2] & [\phi_{U+1},\phi_3] & \dots & [\phi_{U+1},\phi_U] \end{vmatrix}.$$

D is a linear function of the ϕ_β's and so vanishes in the weak sense. The P. b. of D with any quantity f equals the sum of the determinants formed by taking the P. b. of each column of (30) with f. All these determinants, except the one formed by taking the P. b. of the first column with f, will vanish in the weak sense, as the elements of their first column all vanish in the weak sense. Thus

$$(31)\qquad [D, f] = \begin{vmatrix} [\phi_1,f] & 0 & [\phi_1,\phi_2] & [\phi_1,\phi_3] & \dots & [\phi_1,\phi_U] \\ [\phi_2,f] & [\phi_2,\phi_1] & 0 & [\phi_2,\phi_3] & \dots & [\phi_2,\phi_U] \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ [\phi_{U+1},f] & [\phi_{U+1},\phi_1] & [\phi_{U+1},\phi_2] & [\phi_{U+1},\phi_3] & \dots & [\phi_{U+1},\phi_U] \end{vmatrix}.$$

If we take f to be any ϕ_α, the first column of (31) vanishes and so $[D, \phi_\alpha] = 0$. If we take f to be any ϕ_β or χ, the determinant (31) either has two columns identical and so vanishes, or it is a minor of the matrix (29) with U + 1 rows and columns, and vanishes because this matrix is assumed to be of rank U. Thus D has zero P. b. with all the ϕ's and χ's.

It may be that D vanishes in the strong sense on account of the co-factors of the elements of its first column all vanishing in the weak sense. If this is the case, we take a different determinant D, with its columns after the first one corresponding to any U of the columns of (29) and its rows corresponding to any U + 1 of the rows of (29). We can always choose such a determinant D so that the co-factors of the elements of its first column do not all vanish, from the assumption that (29) is of rank U. We get in this way a D which is a first class ϕ and is a linear function of the ϕ_β's. This contradicts the assumption that we had previously put as many ϕ's as possible in the first class.

We can conclude that *if we have put as many ϕ's as possible in the first class, the v's associated with the second class ϕ's all vanish.* The Hamiltonian (20) then reduces to

$$(32)\qquad H \equiv v_\alpha \phi_\alpha,$$

and the general equation of motion (21) becomes

$$(33)\qquad \dot{g} = v_\alpha [g, \phi_\alpha].$$

The vanishing of the v_β's together with the equations (25) ensure that the consistency conditions are all satisfied. The v_a's remain completely arbitrary. Each of them gives rise to a freedom of motion—an arbitrary function in the general solution of the equations of motion. In the standard case there is just one ϕ, which is necessarily first class, and thus there is one arbitrary function in the general solution of the equations of motion. This is connected with the arbitrary character of the independent variable τ.

7. Supplementary conditions. In dealing with a particular dynamical system, we may wish to impose equations on the coordinates and velocities additional to the equations of motion that follow from the Lagrangian. Such supplementary conditions must be introduced as further weak equations in the theory.

With the help of equations (10) (with $\mathfrak{H} \equiv 0$) the supplementary conditions can be expressed as relations between the q's, p's and v's. They may lead to equations between the q's and p's only. Such equations must be treated as extra χ equations, to be joined on to the set (25). They will give rise to further consistency conditions, which are to be handled in the same way as the preceding ones and may lead to still more χ equations. A first class ϕ must now be defined to have zero P. b. also with these new χ's, so the number of first class ϕ's may be reduced by the supplementary conditions. This would cause a reduction in the number of freedoms of motion.

Those of the supplementary conditions that do not give χ equations will give conditions on the v_a variables. These conditions will usually be of a more complicated kind than merely the vanishing of certain v's, like all the conditions on the v's that follow from consistency conditions. They will make a further reduction in the number of freedoms of motion, reducing it to less than the number of first class ϕ's.

8. Transformations of the Hamiltonian form. Take a set of functions $\theta_s(s = 1, 2, \dots \mathrm{s})$ of the q's and p's such that the determinant

$$(34) \qquad \Delta \equiv \begin{vmatrix} 0 & [\theta_1, \theta_2] & [\theta_1, \theta_3] & \dots & [\theta_1, \theta_\mathrm{s}] \\ [\theta_2, \theta_1] & 0 & [\theta_2, \theta_3] & \dots & [\theta_2, \theta_\mathrm{s}] \\ \cdot & \cdot & \cdot & \dots & \cdot \\ [\theta_\mathrm{s}, \theta_1] & [\theta_\mathrm{s}, \theta_2] & [\theta_\mathrm{s}, \theta_3] & \dots & 0 \end{vmatrix}$$

does not vanish in the weak sense. This implies that s must be even. Let $c_{ss'}$ denote the co-factor of $[\theta_s, \theta_{s'}]$ divided by Δ, so that

$$c_{ss'} \equiv - c_{s's}$$

and

$$(35) \qquad c_{ss'}[\theta_s, \theta_{s''}] \equiv \delta_{s's''} \ .$$

Then we can define a new P. b. $[\xi, \eta]^*$ for any two quantities ξ and η by the formula

$$(36) \qquad [\xi, \eta]^* \equiv [\xi, \eta] + [\xi, \theta_s]\, c_{ss'}\, [\theta_{s'}, \eta].$$

It is easily seen that the new P. b.'s obey the first two of the laws (15), and after some calculation one finds that they also obey the third, Poisson's identity. (See Appendix.) The new P. b.'s make

$$
\begin{aligned}
[\xi, \theta_s]^* &\equiv [\xi, \theta_s] + [\xi, \theta_{s'}]\, c_{s's''}[\theta_{s''}, \theta_s] \\
&\equiv [\xi, \theta_s] - [\xi, \theta_{s'}]\, \delta_{s's} \\
&\equiv 0 \qquad (37)
\end{aligned}
$$

for any ξ.

To understand the significance of the new P. b.'s, let us take the case when the θ's consist of $\frac{1}{2}$s coordinates q and their conjugate p's. We then see that the new P. b.'s are obtained by omitting the terms involving differentiations with respect to these q's and p's from the summation over n in the definition (14). Thus the new P. b.'s refer to a system of N $- \frac{1}{2}$s degrees of freedom. If, instead of taking the θ's to be just certain q's and p's, we take them to be any independent functions of these q's and p's, we get the same new P. b.'s. With general θ's the new P. b.'s will still refer to a system of N $- \frac{1}{2}$s degrees of freedom, but the reduction of the degrees of freedom is made in a more complicated way than the mere omission of certain q's and p's.

Let us suppose the θ's are all ϕ's or χ's. (The ϕ's must be second class, as otherwise $\Delta = 0$.) We then have $[\theta_s, H] = 0$ for all s, and hence

$$[g, H]^* = [g, H] = \dot{g} \qquad (38)$$

for g any function of the q's and p's. Thus *the new P. b.'s may be used to give the Hamiltonian equations of motion.* We get in this way a new form for the equations of motion, which is simpler because the number of effective degrees of freedom is reduced.

Each of the θ's now vanishes in the weak sense. If we work only with the new P. b.'s, we can assume each of the θ's vanishes in the strong sense without getting a contradiction, because from (37) the new P. b. of a θ with anything vanishes. We can then use the strong equations $\theta_s \equiv 0$ to simplify the Hamiltonian.

Let us define a χ to be first class it if has zero P. b. with all the ϕ's and χ's and to be second class otherwise. We can make a linear transformation of the χ's of the form

$$\chi^*_k = \gamma_{kk'} \chi_{k'} + \gamma'_{km}\, \phi_m, \qquad (39)$$

where the γ's and γ''s are any functions of the q's and p's such that the determinant of the γ's does not vanish in the weak sense, and the new χ's are then equivalent to the old ones for all the purposes of the theory. Let us make a transformation of this kind so as to bring as many χ's as possible into the first class, and let us call the first class χ's that we then have χ_a's and the second class ones χ_β's.

We may take the θ's to consist of all the ϕ_β's and χ_β's. The determinant Δ then does not vanish. The proof of this result is similar to the proof that the matrix (29) is of rank B, and consists in assuming that Δ is of rank T $<$ S and constructing a determinant like

$$
(40)\qquad \begin{vmatrix} \theta_1 & 0 & [\theta_1, \theta_2] & \cdots & [\theta_1, \theta_T] \\ \theta_2 & [\theta_2, \theta_1] & 0 & \cdots & [\theta_2, \theta_T] \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \theta_{T+1} & [\theta_{T+1}, \theta_1] & [\theta_{T+1}, \theta_2] & \cdots & [\theta_{T+1}, \theta_T] \end{vmatrix},
$$

which is then seen to be a first class ϕ or χ and is a linear function of the ϕ_β's and χ_β's, so it contradicts the assumption that as many ϕ's and χ's as possible have been put in the first class.

With this choice of θ's we get the maximum simplification of the Hamiltonian equations of motion by this method. We get a new scheme in which all the ϕ_β and χ_β equations are strong equations. We may be able to use these equations to eliminate some of the q's and p's entirely from the theory.

The form of the new scheme is not unique, because the ϕ_β's and χ_β's are not unique. If we merely replace the ϕ_β's and χ_β's by linear functions of themselves, we do not change the final form. We can, however, add to the ϕ_β's any linear functions of the ϕ_a's, and to the χ_β's any linear functions of the ϕ_a's and χ_a's, which does not change Δ or the $c_{ss'}$, but does in general change $[\xi, \eta]^*$, and so the form of the Hamiltonian scheme is altered. The different forms must, of course, be equivalent, as they all give the same equations of motion.

As an application of the above method, let us consider the case of a Lagrangian that does not involve some of the velocities. Suppose L does not involve $\dot{q}_j$ ($j = 1, 2, \ldots, \mathrm{J} < \mathrm{N}$). Then each p_j equals zero in the weak sense and equals a ϕ in the strong sense. Suppose that no linear combination of the p_j's is first class. Then we can take the p_j's to be ϕ_β's. Let us now take half the θ's to be the p_j's and the other half to be suitable second class ϕ's or χ's so that Δ does not vanish. Call these other θ's θ_j. With this choice of θ's one easily sees that the new P. b.'s are just what one would get if one applied the definition (14) to those degrees of freedom for which q is not a q_j, with each p_j reckoned as strongly equal to zero and each q_j reckoned as strongly equal to a function of the other q's and p's given by the equations $\theta_j \equiv 0$. We get in this way a new Hamiltonian scheme (not necessarily with the maximum simplification, as there may be other ϕ_β's and χ_β's not included in the θ's) in which the q_j and p_j do not appear as independent dynamical variables.

The new scheme could be obtained in a more direct way by not counting the q_j's as coordinates right from the beginning, and not introducing momenta conjugate to them at all. Let us see what modifications this would bring into the development of the theory.

Define n so as to take on only those values for which q is not a q_j, i.e. the values $\mathrm{J}+1, \mathrm{J}+2, \ldots, \mathrm{N}$. Then equations (2) and (5) still hold and equation (6) must be replaced by

$$
(41)\qquad \delta H = \dot{q}_n \delta p_n - \partial L/\partial q_n . \delta q_n - \partial L/\partial q_j . \delta q_j,
$$

as we allow the q_j's to vary. We may assume the equations

$$
(42)\qquad \partial L/\partial q_j = 0
$$

as supplementary conditions with this method. Equation (41) then reduces to precisely (6). We can infer that H is of the form (20), where the ϕ_m are functions of the q_n's and p_n's, independent of the q_j's, that vanish on account of equations (2). The remainder of the theory can be developed as before, in terms of ϕ's and χ's that do not involve the q_j's . Those of the ϕ or χ equations that do involve the q_j's can be looked upon as defining the q_j's in terms of the other variables, and play no further role in the theory.

With this form of the theory we have the Lagrangian containing variables q_j that involve momenta. The appearance of momentum variables in the Lagrangian is analogous to the appearance of the velocity variables v_a in the Hamiltonian.

9. The Hamiltonian as starting point. Instead of starting with the Lagrangian and obtaining the Hamiltonian from it, one can start with the Hamiltonian. One begins by assuming certain dynamical variables q_n and p_n $(n = 1, 2, \ldots, N)$, or maybe other dynamical variables between which there are definite P. b. relations satisfying the laws (15), and assuming certain weak equations as ϕ equations connecting them. There is no point in distinguishing ϕ's and χ's with this method. At least one of the ϕ's must be first class, i.e. must have zero P. b. with all the ϕ's, or there can be no consistent motion. One then assumes the Hamiltonian to be a linear function of the first class ϕ's ϕ_a, with new variables v_a as coefficients, and assumes the Hamiltonian equations of motion (17) or (33). The v's can vary arbitrarily with the independent variable τ.

The previous scheme of equations of motion, derived from a Lagrangian and involving possibly χ's as well as ϕ's, is to be looked upon as an example of the present scheme with some of the v's restricted to be zero by supplementary conditions. The ϕ_a's corresponding to these v_a's are then the first class χ's of the previous scheme. Such supplementary conditions, or any supplementary conditions involving the v's, are of no value for the application of the theory to relativistic dynamics given in the next section and cannot be taken over into the quantum theory, so they will not be included in the further work. Supplementary conditions not involving the v's are just ϕ equations.

The P. b. of two first class ϕ's is a first class ϕ, as may be verified in the following way. The P. b. $[\phi_a, \phi_{a'}]$ vanishes weakly and so is strongly equal to a linear function of the ϕ's, these being the only quantities that are weakly zero in the present scheme. We have to show that its P. b. with an arbitrary ϕ is weakly zero. From Poisson's identity

$$[\phi, [\phi_a, \phi_{a'}]] \equiv [[\phi, \phi_a], \phi_{a'}] - [[\phi, \phi_{a'}], \phi_a]. \tag{43}$$

Since ϕ_a is first class, $[\phi, \phi_a]$ vanishes weakly and so is strongly equal to a linear function of the ϕ's, and hence its P. b. with the first class $\phi_{a'}$ vanishes weakly. Similarly the second term on the right-hand side of (43) vanishes weakly, so the required result is proved.

Suppose there are A independent first class ϕ's and M independent ϕ's altogether. In the phase space (the 2N-dimensional space of the q_n and p_n variables) there is a space of (2N − M) dimensions in which all the ϕ equations are satisfied. Call it the $(2N - M)$-space. The state of the dynamical system for a particular τ value is fixed by giving values to the q's and p's satisfying all the ϕ equations and is thus represented by a point P in the $(2N-M)$-space. The motion of the system ensuing from this state is represented by a curve in the $(2N - M)$-space starting from P. On account of the A variables v_a being arbitrary, this curve may start out in any direction in a small space of A dimensions surrounding P. There is one of these small spaces of A dimensions surrounding every point of the $(2N-M)$-space. It will now be shown that *these small spaces are integrable.*

Suppose that for an interval of τ, $\delta\tau = \epsilon_1$, all the v's vanish except $v_{a'}$, which is equal to 1, and that for the following τ interval, $\delta\tau = \epsilon_2$, all the v's vanish except $v_{a''}$, which is equal to 1. Then any function g of the q's and p's is changed at the end of the first interval to

$$g + \epsilon_1[g, \phi_{a'}].$$

It is changed at the end of the second interval, with the accuracy $\epsilon_1\epsilon_2$ but with neglect of $\epsilon_1{}^2$ and $\epsilon_2{}^2$, to

$$g + \epsilon_1[g, \phi_{a'}] + \epsilon_2[g + \epsilon_1[g, \phi_{a'}], \phi_{a''}]. \tag{44}$$

If the two kinds of motion are made in the reverse order, g changes to

$$g + \epsilon_2[g, \phi_{a''}] + \epsilon_1[g + \epsilon_2[g, \phi_{a''}], \phi_{a'}]. \tag{45}$$

The difference between (44) and (45) is, by Poisson's identity,

$$\epsilon_1\epsilon_2[g,[\phi_{a'}, \phi_{a''}]]. \tag{46}$$

It was shown above that $[\phi_{a'}, \phi_{a''}]$ is a first class ϕ, so that (46) is a possible change in g arising from the equations of motion with a suitable choice of the v's, and thus corresponds to a motion in the small A-dimensional space round the starting point. This is the condition for integrability.

If there are supplementary conditions involving the v's, this integrability may get spoiled. Thus the integrability does not necessarily hold for the equations of motion derived from a Lagrangian.

The integration of the small spaces will provide a set of A-dimensional spaces lying in the $(2N-M)$-space such that the motion always takes place entirely in one of them. Call these spaces A-spaces. Every curve in an A-space represents a possible solution of the equations of motion. Every point of the $(2N-M)$-space lies in an A-space, which contains all the motions starting from that point. It would be permissible to consider the A-space itself as the complete solution of the equations of motion, rather than a general curve in it.

Given a particular A-space, we can fix a point of it by A coordinates, each of which is some function of the q's and p's. Call these coordinates t_a ($a = 1, 2, \ldots, $ A). They will play the role of time variables. The A-space itself can be described by giving all the q's and p's as functions of the t_a. If g is any q or p, or a function of the q's and p's, we have

(47) $$\dot{g} = \dot{t}_a \, \partial g/\partial t_a,$$

since the τ variation of g may be looked upon as arising from the τ variation of the t_a's. Using the Hamiltonian equations of motion (33) for $\dot{g}$ and $\dot{t}_a$, we get

$$v_a[g, \phi_a] = v_a[t_a, \phi_a] \, \partial g/\partial t_a.$$

This equation holds for arbitrary v_a, so

(48) $$[g, \phi_a] = [t_a, \phi_a] \, \partial g/\partial t_a.$$

Equations (48) may be looked upon as the general equations of motion that fix an A-space. They are the closest equations in the theory with homogeneous velocities to the usual Hamiltonian equations of motion. If A = 1, we may take the one variable t_a to be the time and (48) then reduces to precisely the usual Hamiltonian equations of motion.

To pass from the Hamiltonian to the Lagrangian, we introduce the velocities $\dot{q}_n$ by the equations

(49) $$\dot{q}_n = v_a \, \partial\phi_a/\partial p_n,$$

and then define L by

(50) $$L \equiv p_n\dot{q}_n - H \equiv p_n\dot{q}_n - v_a\phi_a.$$

This gives L as a function of the q's, $\dot{q}$'s, p's and v's, linear in the $\dot{q}$'s and v's. Making independent variations δq, $\delta\dot{q}$, δp, δv, we get

$$\delta L = \dot{q}_n\delta p_n + p_n\delta\dot{q}_n - \phi_a\delta v_a - v_a(\partial\phi_a/\partial q_n.\delta q_n + \partial\phi_a/\partial p_n.\delta p_n)$$

(51) $$= p_n\delta\dot{q}_n - v_a \, \partial\phi_a/\partial q_n.\delta q_n.$$

Thus δL does not depend on the δp_n and δv_a. This result is to be compared with (6).

If the equations (49) together with the ϕ equations give the $\dot{q}$'s as independent functions of the p's and v's, so that they allow the p's and v's to be considered as functions of the q's and $\dot{q}$'s, then (51) shows that L is strongly equal to a function of the q's and $\dot{q}$'s only. This function must be homogeneous of the first degree in the $\dot{q}$'s. Differentiating it partially with respect to a q or $\dot{q}$, we find

(52) $$\begin{aligned} \partial L/\partial\dot{q}_n &= p_n \\ \partial L/\partial q_n &= -v_a \, \partial\phi_a/\partial q_n = \dot{p}_n. \end{aligned}$$

These are the usual Lagrangian equations.

If the equations (49) together with the ϕ equations do not give the $\dot{q}$'s as independent functions of the p's and v's, they lead to certain equations between the q's and $\dot{q}$'s only, say

(53) $$R_j(q, \dot{q}) = 0, \qquad j = 1, 2, \ldots, \text{J}.$$

The R's are homogeneous in the $\dot{q}$'s and we arrange them to be of the first degree. We now proceed by a method analogous to that of §3 with the role of p's and $\dot{q}$'s interchanged. We obtain a result analogous to (9),

$$L \equiv \mathfrak{L} + \lambda_j R_j, \tag{54}$$

where $\mathfrak{L}$ is a function of the q's and $\dot{q}$'s only, which must be homogeneous of the first degree in the $\dot{q}$'s, and the coefficients λ_j are functions of the q's, p's and v's.

We have again equations (52) if the λ's are counted as independent variables in the partial differentiations of L, and L is then homogeneous of the first degree in the $\dot{q}$'s. Thus we have a Lagrangian containing momentum variables of the type considered at the end of the preceding section, with the previous q_j corresponding to the present λ_j and the supplementary conditions (42) giving the equations (53).

10. Application to relativistic dynamics. In the ordinary non-relativistic dynamics one works with the state of a dynamical system at a particular instant of time, this state being specified by giving values to the q's and p's. One has equations of motion which enable one, given the state at one instant, to calculate the state at another instant. These equations of motion, written in the Hamiltonian form with homogeneous velocities, need only one first class ϕ.

To get a dynamical theory which satisfies restricted relativity, we must set up a scheme of equations which applies equally to observers with all velocities. If we work with instants, we must include instants with respect to all observers. An instant is then any flat three-dimensional surface in space-time having a normal in a direction within the light-cone. A general instant needs four parameters to describe it, three to fix the direction of the normal, or the velocity of the observer, and the fourth to distinguish different instants for the same observer.

A relativistic dynamics that involves instants must enable one, given the state at any of these instants, to calculate the state at any other. We must have equations of motion showing how the dynamical variables vary as the instant varies. We can allow the instant to vary arbitrarily, with a translational motion in space-time as well as the direction of its normal varying, and the equations of motion must always apply. Thus *we need four first class ϕ's to give rise to the four freedoms of motion of the instant.*

The four parameters that describe the instant are to be treated as q's, subject to the equations of motion (17) or (33) along with the other q's and p's. They are distinguished from the other q's and p's in that it is specially convenient to take them as the t variables of the equations of motion (48). These equations then show directly how any q or p varies for a given variation of the instant.

There are other forms of relativistic dynamics not involving instants, which have been discussed by the author [2]. There is the point form, in which a state is defined with reference to a point in space-time. This form also needs four first class ϕ's, corresponding to the four freedoms of motion of the point. Then there is the front form, which needs three first class ϕ's, corresponding to the three freedoms of motion of a front. Finally, we may take a state to be defined on a general three-dimensional space-like surface in space-time. There must then be infinitely many first class ϕ's, corresponding to all the deformations that may be made in such a surface. With each of these forms the variables that describe the point, front, or general space-like surface are to be treated as q's, subject to the equations of motion (17) or (33), and are specially convenient to be taken as the t variables of equations (48).

The first class ϕ's discussed above are the fewest with which one can construct a relativistic dynamics in the respective forms. There may be additional ones. For example, an electrodynamics which allows gauge transformations to be made after one has fixed the initial values of all the q's and p's must contain extra freedoms of motion, which will need extra first class ϕ's to give rise to them.

11. Quantization. In order to quantize a dynamical system which one has worked out in the classical theory, one must set up a scheme of linear operators corresponding to the classical dynamical variables q and p, and to functions of them. There are no operators corresponding to the classical variables v, or to velocity variables in general, or to anything involving τ. The operators all operate on the vectors ψ of a Hilbert space, whose representatives in any representation are the wave functions which specify states in the quantum theory. Real classical variables correspond to self-adjoint operators.

The linear operators must be analogous to their classical counterparts in accordance with two general principles. Using the same letter to denote two things that are counterparts, the principles are

(i) P. b. relations between the classical variables correspond to commutation relations between the operators, according to the formula

$$[\xi, \eta] \quad \text{corresponds to} \quad 2\pi(\xi\eta - \eta\xi)/ih.$$

(ii) Weak equations between the classical variables correspond to linear conditions on the vectors ψ, according to the formula

$$X(q, p) = 0 \quad \text{corresponds to} \quad X\psi = 0.$$

The procedure of passing from the classical to the quantum theory is not mathematically well-defined, because whenever a classical quantity involves a product of two factors whose P. b. does not vanish, there is an ambiguity in the order in which the two factors should appear in the corresponding quantum expression. In practice with simple examples one finds no difficulty in deciding what the order should be. With complicated examples it may be impossible to choose the order in each case so as to make all the quantum equations consistent, and then one would not know how to quantize the theory. The present-day methods of quantization are all of the nature of practical rules, whose application depends on considerations of simplicity.

There are certain general features of the passage to the quantum theory that one must pay attention to, in order that the consistency of the quantum equations shall not go wrong in an elementary way. We have in the classical theory a number of ϕ equations (counting χ equations also as ϕ equations), which are to be used in the quantum theory according to the principle (ii). We can transform the classical ϕ's linearly by the transformation (27) and the new ϕ's are just as good as the old ones. If we make the corresponding transformation in the quantum theory, we must take care to put the coefficients γ all to the left of the ϕ's. *A general ϕ in the quantum theory is a linear function of the given ϕ's with coefficients on the left.*

From two quantum equations obtained from ϕ equations by principle (ii)

$$\phi_1\psi = 0, \qquad \phi_2\psi = 0,$$

we can infer

$$\phi_2\phi_1\psi = 0, \qquad \phi_1\phi_2\psi = 0,$$

and hence from principle (i),

$$[\phi_1, \phi_2,]\, \psi = 0.$$

This corresponds to the classical weak equation

$$[\phi_1, \phi_2] = 0.$$

We can infer that *all the ϕ's must be first class if the passage to the quantum theory is possible.*

Given a classical theory with second class ϕ's, one can get a quantum theory from it by first applying a transformation of the type described in §8, which converts all the ϕ_β equations into strong equations. The strong equations will correspond in the quantum theory to equations between operators, which serve to define some of them in terms of the others.

The quantum equations $\phi\psi = 0$, obtained by applying principle (ii) to the first class ϕ equations of the classical theory, are the Schroedinger wave equations. The usual classical dynamics with only one first class ϕ leads to only one Schroedinger equation. In the general theory there is one Schroedinger equation for each classical freedom of motion. The operators in these equations all correspond to classical dynamical variables for one τ value. The operators referring to a different τ value do not belong to the same algebraic scheme, and there does not seem to be anything in the quantum theory analogous to the τ dependence of the classical variables.

However, the dependence of the classical variables on the parameters t, given by equations (48), does have a quantum analogue, provided the t's are chosen so as to have zero P. b.'s with one another, so that they can be given numerical values simultaneously in the quantum theory. The specially convenient t's for the various forms of relativistic dynamics discussed in the preceding section do satisfy this condition. We cannot immediately take over equations (48) into the quantum theory because, as easily verified, the equations that we should get would not be invariant under a general linear transformation of the ϕ's (27). We must first put equations (48) in a standard form. By a transformation (27) we introduce a new set of ϕ's, ϕ_a say, in one-one correspondence with the t_a's, so that

$$[t_a, \phi_{a'}] = \delta_{aa'}. \tag{55}$$

With these ϕ's equations (48) reduce to

$$[g, \phi_a] = \partial g/\partial t_a. \tag{56}$$

These equations can be taken over into the quantum theory, and are then Heisenberg's quantum equations of motion for the present generalized dynamics.

Appendix. Proof of Poisson's identity for the new P. b.'s defined by (36).

Use the suffixes $r, s, t, \ldots$ to distinguish different θ's. We have by the definition

$$\begin{aligned} &[[\xi,\eta]^*, \zeta]^* \\ &\quad = [[\xi,\eta] + [\xi,\theta_r]c_{rs}[\theta_s,\eta],\zeta] + [[\xi,\eta] + [\xi,\theta_r]c_{rs}[\theta_s,\eta], \theta_t]c_{tu}[\theta_u,\zeta] \\ &\quad = [[\xi,\eta], \zeta] + [[\xi,\theta_r], \zeta]c_{rs}[\theta_s,\eta] + [\xi,\theta_r][c_{rs},\zeta][\theta_s,\eta] + [\xi,\theta_r]c_{rs}[[\theta_s,\eta],\zeta] \\ &\qquad + [[\xi,\eta], \theta_t]c_{tu}[\theta_u,\zeta] + [[\xi,\theta_r], \theta_t]c_{rs}[\theta_s,\eta]c_{tu}[\theta_u,\zeta] \\ &\qquad + [\xi,\theta_r][c_{rs},\theta_t][\theta_s,\eta]c_{tu}[\theta_u,\zeta] + [\xi,\theta_r]c_{rs}[[\theta_s,\eta], \theta_t]c_{tu}[\theta_u,\zeta]. \end{aligned} \tag{57}$$

Let the operator $\sum$ denote the application of the three cyclic permutations of ξ, η, ζ and the summation of the three results. Then we have to prove that

$$\sum[[\xi,\eta]^*,\zeta]^* = 0.$$

$\sum$ applied to the first term of (57) gives zero, from the ordinary Poisson's identity. $\sum$ applied to the second, fourth and fifth terms gives

$$\sum c_{rs}[\theta_s,\eta]\{[[\xi,\theta_r], \zeta] + [[\theta_r,\zeta],\xi] + [[\zeta,\xi],\theta_r]\} = 0,$$

from the ordinary Poisson's identity again. $\sum$ applied to the sixth and eighth terms of (57) gives, with a cyclic permutation of r, u, s, t in the latter,

$$c_{rs}c_{tu}\sum[\theta_s,\eta][\theta_u,\zeta]\{[[\xi,\theta_r],\theta_t] + [[\theta_t,\xi], \theta_r]\} = -c_{rs}c_{tu}\sum[\theta_s,\eta][\theta_u,\zeta][[\theta_r,\theta_t],\xi]. \tag{58}$$

From (35) we can infer

$$[c_{tu}[\theta_r,\theta_t],\xi] = 0$$

or

$$[c_{tu},\xi][\theta_r,\theta_t] + c_{tu}[[\theta_r,\theta_t],\xi] = 0. \tag{59}$$

Thus (58) reduces to

$$c_{rs}[\theta_r,\theta_t]\sum[\theta_s,\eta][\theta_u,\zeta][c_{tu},\xi] = \sum[\theta_t,\eta][\theta_u,\zeta][c_{tu},\xi],$$

with a further use of (35). This cancels with $\sum$ applied to the third term of (57). $\sum$ applied to the remaining term of (57), the seventh, gives

$$[\xi,\theta_r][\eta,\theta_s][\zeta,\theta_u]\{c_{tu}[c_{rs},\theta_t] + c_{tr}[c_{su},\theta_t] + c_{ts}[c_{ur},\theta_t]\}. \tag{60}$$

Using $\sum_{rsu}$ to denote the operation of applying the three cyclic permutations of r, s, u and simultaneously of r', s', u' and adding the three results, we have from the ordinary Poisson's identity

$$\sum_{rsu} c_{r'r} c_{s's} c_{u'u} [[\theta_{r'}, \theta_{s'}], \theta_{u'}] = 0. \tag{61}$$

From (59) with ξ replaced by $\theta_{u'}$.

$$[c_{r'r}, \theta_{u'}][\theta_{r'}, \theta_{s'}] + c_{r'r}[[\theta_{r'}, \theta_{s'}], \theta_{u'}] = 0,$$

so (61) gives

$$\sum_{rsu} c_{s's} c_{u'u} [\theta_{r'}, \theta_{s'}][c_{r'r}, \theta_{u'}] = 0.$$

With the help of (35), this reduces to

$$\sum_{rsu} c_{u'u} [c_{rs}, \theta_{u'}] = 0,$$

which shows that (60) vanishes. This completes the proof. All the above equations may be written as strong equations, as no weak equations are used in the proof.

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