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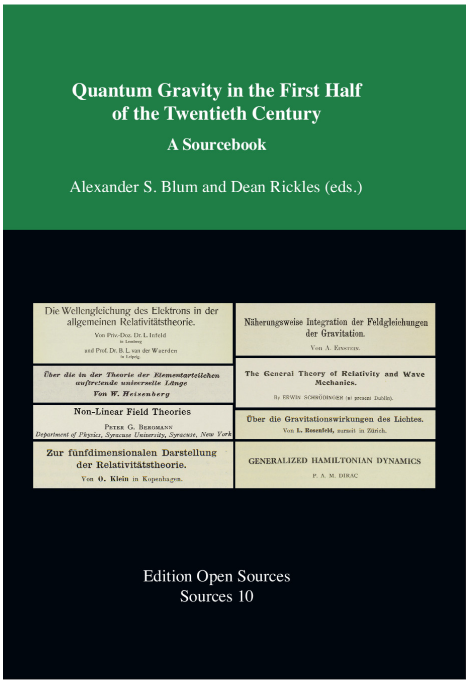

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Chapter 4
George B. Jeffery (1921): The Field of an Electron on Einstein's Theory of Gravitation

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The Field of an Electron on Einstein's Theory of Gravitation.

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(Communicated by Prof. L. N. G. Filon, F.R.S. Received December 16, 1920,—
Revised January 11, 1921.)

Einstein's Generalised Theory of Relativity has accomplished notable results in the region of astronomical mechanics, but for the moment it seems difficult to go further in this direction until some further progress is made with the theory of the solution of the field equations. Practically all the consequences of the theory which have been established so far are obtained from the solution of the equations corresponding to a single isolated singularity. The *exact* solutions corresponding to two isolated singularities are urgently required for the further exploration of the theory. The approximate solutions which have been put forward by De Sitter,* Dröste† and Einstein,‡ though valuable in default of exact solutions, are apt to be misleading and perhaps to exclude effects of far-reaching theoretical importance such as radiation. Meanwhile it is well to examine the consequences of the theory at the other extreme of the realm of physical science, in its relation to atomic phenomena. Here it seems no longer justifiable to consider gravitation and electricity separately. In these microscopic phenomena, where we are free from the effects of averaging, mass and charge seem to be inextricably connected. They exist in certain definite combinations, possibly in only two combinations as electrons and hydrogen nuclei. The gravitational field corresponding to such a charged particle has been investigated by Nordström§ by the application of the calculus of variations to the Hamiltonian function of the combined fields. Nordström's work was not brought to our notice until after the present paper was written and we had obtained the same result by direct solution of the field equations. As the methods employed may perhaps be more familiar to English students of the subject it has been thought well to give a brief outline of this proof as an alternative to Nordström's in the first section of this paper.

* 'Monthly Notices of the Royal Astronomical Society,' vol. 72, p. 155 (1916).

† 'Proc. Acad. Amsterdam,' vol. 19, p. 447 (1916).

‡ 'Berlin, Sitzungsberichte,' 1916, p. 688.

§ "On the Energy of the Gravitational Field in Einstein's Theory," 'Proc. Ac. Amsterdam,' vol. 20, p. 1236 (1918).

§ 1. *The Solution of the Field Equations.*

The gravitational field is defined by the symmetrical covariant tensor $g_{\mu\nu}$, where

$$ds^2 = \Sigma\Sigma\,(\mu,\,\nu)\,g_{\mu\nu}\,dx_\mu\,dx_\nu \qquad (\mu,\,\nu = 1,\,2,\,3,\,4).$$

where the variables of summation are enclosed in brackets immediately following the signs of summation.

For a symmetrical static field we may, following Eddington* and Schwartzchild, take polar co-ordinates r, θ, ϕ, $\tau = ct$ and

$$ds^2 = -e^\lambda\,dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2 + e^\nu\,d\tau^2, \tag{1}$$

where λ, ν are functions of r only, so that $g_{\mu\nu} = 0$ if $\mu \neq \nu$ and

$$g_{11} = -e^\lambda, \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta, \quad g_{44} = e^\nu, \tag{2}$$

The determinant of $g_{\mu\nu}$ is given by

$$g = -r^4\sin^2\theta\,e^{\lambda+\nu}. \tag{3}$$

and the associated contravariant tensor by

$$g^{\mu\mu} = 1/g_{\mu\mu} \qquad \text{and} \qquad g^{\mu\nu} = 0 \text{ for } \mu \neq \nu. \tag{4}$$

The Christoffel symbols, defined by

$$\{\lambda\mu,\,\nu\} = \tfrac{1}{2}\,\Sigma\,(\alpha)\,g^{\nu\alpha}\left(\frac{\partial g_{\lambda\alpha}}{\partial x_\mu} + \frac{\partial g_{\mu\alpha}}{\partial x_\lambda} - \frac{\partial g_{\lambda\mu}}{dx_\alpha}\right),$$

are then as given by Eddington,† accents denoting differentiation with respect to r,

$$\left.\begin{aligned}
&\{11,\,1\} = \tfrac{1}{2}\lambda', \qquad \{12,\,2\} = \{13,\,3\} = 1/r, \qquad \{14,\,4\} = \tfrac{1}{2}\nu', \\
&\{22,\,1\} = -re^{-\lambda}, \qquad \{23,\,3\} = \cot\theta, \\
&\{33,\,1\} = -r\sin^2\theta e^{-\lambda}, \qquad \{33,\,2\} = -\sin\theta\cos\theta, \\
&\{44,\,1\} = \tfrac{1}{2}\nu' e^{\nu-\lambda},
\end{aligned}\right\}, \tag{5}$$

the rest vanishing identically.

The components of Einstein's contracted tensor $G_{\mu\nu}$ vanish with the exception of

$$\left.\begin{aligned}
G_{11} &= \tfrac{1}{2}\nu'' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 - \lambda'/r. \\
G_{22} &= e^{-\lambda}\{1 + \tfrac{1}{2}r(\nu' - \lambda')\} - 1. \\
G_{33} &= \sin^2\theta\,[e^{-\lambda}\{1 + \tfrac{1}{2}r(\nu' - \lambda')\} - 1] \\
G_{44} &= -e^{\nu-\lambda}\{\tfrac{1}{2}\nu'' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 + \nu'/r\}.
\end{aligned}\right\}. \tag{6}$$

* 'Report on Relativity to the Physical Society,' second edition, p. 44. Eddington's Methods have been closely followed in this paper, and references to this source will be quoted as 'Report.'

† 'Report,' p. 45.

The electromagnetic field is defined* by the covariant vector

$$\kappa_\mu \equiv (-\mathrm{F},\ -\mathrm{G},\ -\mathrm{H},\ \Phi)$$

where F, G, H, are the components of the vector potential and Φ is the scalar potential. For static symmetry about the origin $n_2 = n_3 = 0$, and n_1, n_4 are functions of r only.

Denoting the covariant derivative of κ_μ by $\kappa_{\mu\sigma}$ the electromagnetic covariant tensor is defined by

$$\mathrm{F}_{\mu\sigma} = \kappa_{\mu\sigma} - \kappa_{\sigma\mu},$$

and in our case all the components vanish with the exception of

$$\mathrm{F}_{14} = -\kappa_4', \qquad \mathrm{F}_{41} = \kappa_4', \tag{7}$$

the accents as before denoting differentiation with respect to r.

The associated mixed and contravariant tensors are defined respectively by

$$\mathrm{F}_\mu{}^\sigma = \Sigma\,(\alpha)\, g^{\sigma\alpha}\,\mathrm{F}_{\mu\alpha}, \qquad \mathrm{F}^{\mu\sigma} = \Sigma\Sigma\,(\alpha,\,\beta)\, g^{\mu\alpha} g^{\sigma\beta}\,\mathrm{F}_{\alpha\beta}, \tag{8}$$

and in our case the only surviving components are

$$\mathrm{F}_4{}^1 = g^{11}\mathrm{F}_{41} = -e^{-\lambda}\kappa_4', \qquad \mathrm{F}_1{}^4 = g^{44}\mathrm{F}_{14} = -e^{-\nu}\kappa_4' \tag{9}$$

and

$$\mathrm{F}^{14} = g^{11} g^{44}\,\mathrm{F}_{14} = e^{-(\lambda+\nu)}\,\kappa_4' = -\mathrm{F}^{41}. \tag{10}$$

In the absence of charge, Maxwell's equations of the electromagnetic field are expressed† by the vanishing of the contracted covariant derivative of $\mathrm{F}^{\mu\sigma}$,

$$\mathrm{F}_\sigma{}^{\mu\sigma} \equiv \frac{1}{\sqrt{(-g)}}\,\Sigma\,(\sigma)\,\frac{\partial}{\partial x_\sigma}\,[\sqrt{(-g)}\,\mathrm{F}^{\mu\sigma}]$$

in Lorentz units.

These lead to a single equation in n_4,

$$\frac{\partial}{\partial r}\,(r^2\kappa_4' e^{-\frac{1}{2}(\lambda-\nu)}) = 0. \tag{11}$$

The mixed electromagnetic energy tensor is defined by

$$\mathrm{E}_\sigma{}^\mu = -\Sigma\,(\alpha)\,\mathrm{F}_{\sigma\alpha}\,\mathrm{F}^{\mu\alpha} + \tfrac{1}{4} g_\sigma{}^\mu\,\Sigma\Sigma\,(\alpha,\,\beta)\,\mathrm{F}^{\alpha\beta}\,\mathrm{F}_{\alpha\beta}.$$

In our case the only surviving components are

$$\left.\begin{aligned} \mathrm{E}_1{}^1 &= \mathrm{E}_4{}^4 = \tfrac{1}{2}\kappa_4'^2\, e^{-(\lambda+\nu)} \\ \mathrm{E}_2{}^2 &= \mathrm{E}_3{}^3 = -\tfrac{1}{2}\kappa_4'^2\, e^{-(\lambda+\nu)} \end{aligned}\right\}. \tag{12}$$

The associated covariant energy tensor

$$\mathrm{E}_{\mu\sigma} = \Sigma\,(\alpha)\, g_{\alpha\sigma}\,\mathrm{E}_\mu{}^\alpha$$

vanishes except for the components

$$\left.\begin{aligned} \mathrm{E}_{11} &= -\tfrac{1}{2}\kappa_4'^2 e^{-\nu}, & \mathrm{E}_{22} &= \tfrac{1}{2} r^2\kappa_4'^2 e^{-(\lambda-\nu)} \\ \mathrm{E}_{33} &= \tfrac{1}{2} r^2 \sin^2\theta\,\kappa_4'^2 e^{-(\lambda+\nu)}, & \mathrm{E}_{44} &= \tfrac{1}{2}\kappa_4'^2 e^{-\lambda} \end{aligned}\right\}, \tag{13}$$

* 'Report,' p. 76.

† 'Report,' p. 77.

and the associated scalar

$$E = \Sigma\Sigma(\mu\sigma) g^{\mu\sigma} E_{\mu\sigma} = \Sigma(\alpha, \mu) g_{\alpha}{}^{\mu} E_{\mu}{}^{\alpha} = E_1{}^1 + E_2{}^2 + E_3{}^3 + E_4{}^4 = 0. \quad (14)$$

Omitting the terms representing the density and motion of matter, Einstein's equations of the gravitational field are

$$G_{\mu\sigma} = -8\pi\kappa c^{-4}(E_{\mu\sigma} - \tfrac{1}{2} g_{\mu\sigma} E),$$

where κ is the constant of gravitation and c the velocity of light

These give

$$G_{11} = 4\pi\kappa c^{-4}\kappa_4'^2 e^{-\nu},$$

$$G_{22} = -4\pi\kappa c^{-4} r^2 \kappa_4'^2 e^{-(\lambda+\nu)},$$

$$G_{33} = -4\pi\kappa c^{-4} r^2 \sin^2\theta\, \kappa_4'^2 e^{-(\lambda+\nu)},$$

$$G_{44} = -4\pi\kappa c^{-4}\kappa_4'^2 e^{-\lambda},$$

the remaining components of $G^{\mu\sigma}$ vanishing.

Hence from (6) and (15) we have

$$\tfrac{1}{2}\nu'' - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 - \lambda' r = 4\pi\kappa c^{-4}\kappa_4'^2 e^{-\nu}, \quad (16)$$

$$e^{-\lambda}\{1 + \tfrac{1}{2} r(\nu' - \lambda')\} - 1 = -4\pi\kappa c^{-4} r^2 \kappa_4'^2 e^{-(\lambda+\nu)}, \quad (17)$$

$$\sin^2\theta\,[e^{-\lambda}\{1 + \tfrac{1}{2} r(\nu' - \lambda')\} - 1] = -4\pi\kappa c^{-4} r^2 \sin^2\theta\,\kappa_4'^2 e^{-(\lambda+\nu)}, \quad (18)$$

$$\tfrac{1}{2}\nu'^1 - \tfrac{1}{4}\lambda'\nu' + \tfrac{1}{4}\nu'^2 + \nu'/r = 4\pi\kappa c^{-4}\kappa_4'^2 e^{-\nu}. \quad (19)$$

Equations (17) and (18) are clearly identical, while (16) and (19) give $\lambda' + \nu' = 0$ or $\lambda + \nu = const.$ If the gravitational field disappears at a great distance from the origin we may take $\lambda + \nu = 0$.

Equation (11) then gives

$$\kappa_4 = \frac{\epsilon}{4\pi r}, \quad (20)$$

where ϵ is a constant which will presently be identified with the electric charge in Lorentz units, and a second constant of integration has been put equal to zero without loss of generality.

Equation (17) then gives

$$e^{\nu}(1 + r\nu') = 1 - \frac{\kappa\epsilon^2}{4\pi c^4 r^2},$$

or writing $e^{\nu} = \gamma$,

$$\gamma = 1 - \frac{2\kappa m}{c^2 r} + \frac{\kappa\epsilon^2}{4\pi c^4 r^2}, \quad (21)$$

where m is a constant of integration which will subsequently be seen to be the mass. It may now be verified that equations (16) and (19) are satisfied identically.

We have therefore as the line element of the space surrounding a point which is a singularity of *both* the electric and gravitational fields

$$ds^2 = -\gamma^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + \gamma c^2 dt^2, \quad (22)$$

which is the result given by Nordström.

§2. *The Motion of a Particle in the Field of a Charged Nucleus.*

We will investigate the motion of a particle having both charge and inertia mass in the field which we have examined in the previous section. The mass and charge of the particle will be supposed so small that, while the motion of the particle is determined by the field, the field itself is not appreciably affected by the motion of the particle. This is a serious limitation, but it is the best that we can do in default of the solution corresponding to two singularities. It neglects all effects due to radiation and to the reaction upon the nucleus. If the nucleus is itself an electron, it is difficult to imagine any physical phenomenon to which the analysis can apply, unless indeed we regard the electron as composed of still smaller elements in a disposition which is stable for the particular values of the electronic charge, mass and radius. The analysis will, however, give a good approximation to the motion of an electron in the neighbourhood of an atomic nucleus, for then the mass of the nucleus is at least about a thousand times that of the electron, and, while the charges may be of the same order, it will appear that the terms in $g_{\mu\nu}$ which depend on the charge are in all practical cases small compared with those depending upon the mass.

The "force" acting on a charged particle is given by* the covariant vector

$$k_\sigma = \mathrm{F}_{\sigma\mu}\,\mathrm{J}^\mu,$$

where J^μ is the charge-current vector which, for a single particle of charge ϵ' is

$$\epsilon'\left(\frac{dx_1}{ds},\ \frac{dx_2}{ds},\ \frac{dx_3}{ds},\ \frac{dx_4}{ds}\right).$$

The associated contravariant vector is

$$k^\sigma = \Sigma\,(\alpha)\,g^{\alpha\sigma}\,k_\alpha. \tag{23}$$

The contravariant acceleration vector† is

$$\mathrm{A}^\sigma = \frac{d^2x_\sigma}{ds^2} + \Sigma\Sigma\,(\alpha,\beta)\,\{\alpha\beta,\sigma\}\,\frac{dx_\alpha}{ds}\,\frac{dx_\beta}{ds}. \tag{24}$$

We will take as the equations of motion of our particle

$$m'\,\mathrm{A}^\sigma = -\frac{1}{c^2}\,k^\sigma, \tag{25}$$

where m' is the mass of the particle.

In rectangular co-ordinates $x, y, z, \tau = ct$ in a non-gravitational field, the Christoffel symbols vanish, and if v is the velocity of the particle

$$\frac{d}{ds} = \frac{1}{\sqrt{(c^2-v^2)}}\,\frac{d}{dt}.$$

* 'Report,' p. 78.

† 'Report,' p. 48.

Remembering that the electric and magnetic vectors (E, B) are given by

$$\mathrm{B} = curl\,(\mathrm{F}, \mathrm{G}, \mathrm{H}), \qquad \mathrm{E} = -grad\Phi - \frac{1}{c}\frac{\partial}{\partial t}(\mathrm{F}, \mathrm{G}, \mathrm{H}),$$

the equations (25) are easily reduced to

$$\frac{d}{dt}\left\{\frac{m'}{\sqrt{(1-v^2/c^2)}}\frac{dx}{dt}\right\} = \epsilon'\left(\mathrm{E}_x + \frac{1}{c}(v_y \mathrm{B}_z - v_z \mathrm{B}_y)\right)$$

and two similar equations, with

$$\frac{d}{dt}\left\{m'c^2\left(\frac{1}{\sqrt{(1-v^2/c^2)}}-1\right)\right\} = \epsilon'\,(\mathrm{E}_x v_x + \mathrm{E}_y v_y + \mathrm{E}_z v_z).$$

These are the equations of motion on the restricted relativity theory, and reduce to the classical equations for small velocities. Equations (25), therefore, hold in the absence of gravitation, and being of the necessary tensor form they will, by the equivalence hypothesis, be true for any system of co-ordinates and in a gravitational field.

In our co-ordinates

$$k_1 = \frac{\epsilon\epsilon' c}{4\pi r^2}\frac{dt}{ds}, \quad k_2 = k_3 = 0, \quad k_4 = -\frac{\epsilon\epsilon'}{4\pi r^2}\frac{dr}{ds}$$

and hence

$$k^1 = -\frac{\epsilon\epsilon' c e^{\nu}}{4\pi r^2}\frac{dt}{ds}, \quad k^2 = k^3 = 0, \quad k^4 = -\frac{\epsilon\epsilon' e^{-\nu}}{4\pi r^2}\frac{dr}{ds}.$$

Substituting from (5) and (24) in (25) we have

$$\frac{d^2 r}{ds^2} - \tfrac{1}{2}\nu'\left(\frac{dr}{ds}\right)^2 - re^{\nu}\left(\frac{d\theta}{ds}\right)^2 - r\sin^2\theta\, e^{\nu}\left(\frac{d\phi}{ds}\right)^2 + \tfrac{1}{2}c^2\nu' e^{2\nu}\left(\frac{dt}{ds}\right)^2 = \frac{\epsilon\epsilon' e^{\nu}}{4\pi c m' r^2}\frac{dt}{ds}, \quad (26)$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \quad (27)$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\,\frac{d\theta}{ds}\frac{d\phi}{ds} = 0, \quad (28)$$

$$\frac{d^2 t}{ds^2} + \nu'\frac{dr}{ds}\frac{dt}{ds} = \frac{\epsilon\epsilon' e^{-\nu}}{4\pi c^3 m' r^2}\frac{dr}{ds}. \quad (29)$$

For motion in the equatorial plane we may put $\theta = \frac{1}{2}\pi$, and (27) is then identically satisfied. From (28) and (29) we then have

$$r^2\frac{d\phi}{ds} = h, \qquad \gamma\frac{dt}{ds} = n - \frac{\epsilon\epsilon'}{4\pi c^3 m' r}, \quad (30)$$

where h and n are constants. Eliminating ϵ, ϵ' between (26) and (29) we obtain a third integral

$$\frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - c^2\gamma\left(\frac{dt}{ds}\right)^2 + 1 = 0, \quad (31)$$

where γ is given by (21).

Considering a particle at rest in three dimensional space so that $\frac{dr}{ds} = \frac{d\phi}{ds} = 0$, we obtain from (26) and (31)

$$m' \frac{d^2r}{dt^2} = -\kappa \frac{mm'}{r^2} + \frac{\epsilon\epsilon'}{4\pi r^2}$$

where terms of higher order than r^{-2} have been neglected. Hence this is the equation of motion of a particle at great distances from the origin, and our interpretation of the constants m, ϵ as the mass and charge of the nucleus respectively, is confirmed.

We have used Lorentz units in order to maintain symmetry among the components of the various tensors employed. We may now revert to ordinary electrostatic units by omitting the factor 4π in (21) and (30). We then have as our equations of motion,

$$\frac{1}{\gamma}\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\phi}{ds}\right)^2 - c^2\gamma\left(\frac{dt}{ds}\right)^2 + 1 = 0, \qquad (32)$$

with

$$r^2 \frac{d\phi}{ds} = h, \qquad (33)$$

and

$$\gamma \frac{dt}{ds} = n - \frac{\epsilon\epsilon'}{c^3 m' r}, \qquad (34)$$

where

$$\gamma = 1 - \frac{2\kappa m}{c^2 r} + \frac{\kappa\,\epsilon^2}{c^4 r^2}. \qquad (35)$$

If we eliminate t, s from (32) by means of (33) and (34), and write $u = 1/r$, we obtain as the geometrical equation of the orbits

$$h^2\left(\frac{du}{d\phi}\right)^2 = c^2\left(n - \frac{\epsilon\epsilon'}{c^3 m'}u\right)^2 - (1 + h^2u^2)\left(1 - \frac{2\kappa m}{c^2}u + \frac{\kappa\epsilon^2}{c^4}u^2\right). \qquad (36)$$

We will reserve a detailed investigation of these orbits for a future communication. We may, however, point out that it follows from (36) that u, and therefore r, can be expressed as an elliptic function of ϕ. Since the coefficients of (36) are real and the right-hand side is necessarily positive, this elliptic function has a real period. Hence the orbit is periodic in the sense that r is a periodic function of ϕ, although the period will not in general be 2π. The orbit will, therefore, not be closed, and there will be a rotation of the apse lines. This periodicity of the orbit is due to the fact that we have taken no account of the modification in the field due to the moving particle. A more complete solution, which included the effects of both particles, would probably lead to orbits which would not generally be periodic. This would correspond to the fact pointed out by Einstein, that moving masses imply radiation of gravitational energy.

Einstein seems to infer from this that his law of gravitation must be subject to the restrictions of the Quantum theory. He says* :—

"Gleichwohl müssten die Atome zufolge der inneratomischen Elektronenbewegung nicht nur electromagnetische, sondern auch Gravitationsenergie ausstrahlen, wenn auch in winzigem Betrage. Da dies in Wahrheit in der Natur nicht zutreffen dürfte, so scheint es, dass die Quantentheorie nicht nur die Maxwellsche Elektrodynamik, sondern auch die neue Gravitationstheorie wird modifizieren müssen."

Einstein may be correct in his speculation, but is there not another possibility? An investigation of the motion of a particle of infinitesimal mass in the field of a particle of finite mass has led us to equation (36) and orbits which are always strictly periodic. The complete solution of the problem of two bodies might well lead to an equation corresponding to (36) and to orbits which are not in general periodic, but which *may* in certain circumstances be periodic. These would give the "quantised" orbits. If it should then appear that the non-periodic orbits tended to asymptote down to the periodic orbits, and we could trace the radiation emitted in the process, the whole secret of the Quantum hypothesis would be laid bare. This is, of course, mere speculation, but it may serve to show the extreme importance of obtaining the exact solution of Einstein's equations corresponding to *two* point singularities.

§ 3. *The Conception of a Point Electron.*

We observe from (22) that the effect of the singularity upon the gravitational field is given by the deviation of γ from its value at infinity, namely unity. As we approach the singularity from infinity γ steadily decreases to a minimum at $r = \epsilon^2/mc^2$ and then increases and finally reaches a positive infinity at the singularity. It is equal to its value at infinity when

$$r = \epsilon^2/2mc^2. \tag{37}$$

For an electron we may take $\epsilon = 3 \times 10^{-10}$ C.G.S. electrostatic units, $m = 10^{-27}$ grm. and $c = 3 \times 10^{10}$ cm./sec. and (37) then gives $r = 0{\cdot}5 \times 10^{-13}$ which is of the same order as the usually accepted value of the radius of an electron. It would be easy to attach too much importance to this numerical result. Firstly, it is not clear that the equality of γ, and, therefore, of the $g_{\mu\nu}$, at two different places has any precise physical meaning. It probably depends upon the mode of measuring r, which in this theory is largely at our disposal. Secondly, the third term in (35) represents the effect of electromagnetic energy upon the gravitational field and by equating γ to unity we

* 'Berlin Sitzungsberichte,' 1916, p. 696.

have obtained a relation between mass and electromagnetic energy. Such a relation is given by the ordinary electromagnetic theory of mass, and in fact as far as we are aware the only determination of the radius of an electron is by means of a formula for the radius in terms of the mass and charge, which differs from (37) only by a numerical factor which depends upon the distribution of charge in the electron. It is, therefore, not surprising that we should obtain a radius of the same order of magnitude. At the same time the present point of view suggests a new meaning for the "radius" of an electron. On the older theory of electromagnetic mass we were forced to regard an electron as a body of small but finite dimensions; it could not be a mere *point* singularity of the field, as the mass of such a point charge would be infinite.

In the same way we cannot have a point singularity on Einstein's theory if we neglect the effects of electric charge, for in this case $\gamma = 1 - 2\kappa m/c^2 r$ and $\gamma = 0$ for $r = 2\kappa m/c^2$. Hence at $r = 0$, $g_{11} = 0$, $g_{44} = \infty$ and at $r = 2\kappa m/c^2$ $g_{11} = \infty$, $g_{44} = 0$. We have, therefore, not a solution with a single point singularity, but a solution with a point singularity surrounded by a spherical surface of singularity.

On the other hand if we include the effects of charge, we see from (35) that γ has no zeros or infinities other than $r = 0$, if $\epsilon^2/m^2 > \kappa$, *i.e.*, $> 0{\cdot}67 \times 10^{-7}$ C.G.S. units. This condition is amply satisfied, both in the case of the electron and the hydrogen nucleus. We may therefore, if we wish, regard these as true point singularities of the field. As we approach such a point very closely γ becomes very large and the field is profoundly modified. As a measure of the dimensions of this region we may take the radius of a sphere on the surface of which the g's have the same value as at infinity. It is this radius which we have calculated above as the radius of the electron.

§ 4. *The Effect of an Electron upon Radiation in its Field.*

According to Einstein's theory, a ray of light passing near to an attracting mass is deflected, and it appears that the deflection would be very great for a ray passing very close to the attracting mass. Indeed, it would seem to be not impossible that a ray which passed sufficiently close to an attracting particle might be so strongly deflected that it would be permanently entrapped by the particle. If this should prove to be a legitimate deduction from Einstein's theory, it might have far-reaching consequences on our views as to the nature of the electron. It will be shown, however, that if we allow for both the charge and the mass of the electron, even though we regard it as a point singularity, no such result is possible. The deflection is always finite. As the distance of nearest approach diminishes, the deflection increases up to a maximum and then diminishes, becomes negative, and finally approaches $-\pi$, so that the ray is returned very nearly along its original direction.

For the path of a ray of light, we have $ds = 0$, which, for a ray in the equatorial plane $\theta = \frac{1}{2}\pi$, gives

$$\gamma c^2 dt^2 = \gamma^{-1} dr^2 + r^2 d\phi^2. \tag{38}$$

We have also to make the integral $\int dt$ stationary for given initial and final points. From (38), we have

$$\int c\,dt = \int \frac{1}{\gamma}\left\{1+\gamma r^2\left(\frac{d\phi}{dr}\right)^2\right\}^{\frac{1}{2}} dr.$$

Hence

$$\delta \int c\,dt = \int r^2 \frac{d\phi}{dr}\left\{1+\gamma r^2\left(\frac{d\phi}{dr}\right)^2\right\}^{-\frac{1}{2}} \frac{d}{dr}(\delta\phi)\,dr$$

$$= \left| r^2 \frac{d\phi}{dr}\left\{1+\gamma r^2\left(\frac{d\phi}{dr}\right)^2\right\}^{-\frac{1}{2}} \delta\phi \right|$$

$$- \int \frac{d}{dr}\left[r^2 \frac{d\phi}{dr}\left\{1+\gamma r^2\left(\frac{d\phi}{dr}\right)^2\right\}^{-\frac{1}{2}}\right] \delta\phi\,dr.$$

The expression to be taken between limits vanishes, since $\delta\phi$ vanishes at the extreme points of the path, and the integral vanishes if

$$r^4\left(\frac{d\phi}{dr}\right)^2 = p^2\left\{1+\gamma r^2\left(\frac{d\phi}{dr}\right)^2\right\}$$

where p^2 is a constant.

Writing $u = 1/r$, and substituting the value of γ from (35), this may be written

$$\left(\frac{du}{d\phi}\right)^2 = \frac{1}{p^2} - u^2 + \frac{2\kappa m}{c^2} u^3 - \frac{\kappa\epsilon^2}{c^4} u^4, \tag{39}$$

from which it may be shown that p is the length of the perpendicular drawn from the origin on to the asymptotes of the path of the ray.

The right-hand side of (39) is a quartic in u, and hence u may be expressed as an elliptic function of ϕ. This elliptic function could be worked out explicitly if for any reason it should become desirable to investigate the exact form of the rays. The analysis is however heavy, and we can obtain all the information we require by other methods.

Denoting the right-hand side of (39) by $f(u)$, we have

$$f'(u) = -2u\left(1 - \frac{3\kappa m}{c^2} u + \frac{2\kappa\epsilon^2}{c^4} u^2\right).$$

Inserting the values of m and ϵ given in the last section, it will be seen that the quadratic on the right-hand side has no real roots in the case of the electron or the hydrogen nucleus, and, therefore, the only real root of

$f'(u) = 0$ is $u = 0$. Hence $f(u) = 0$ has at most two real roots. It then appears from (39) that, for real rays $p^2 > 0$, so that all real rays have real asymptotes, and it is not possible for light to circle in a closed "light orbit" round the electron. If $p^2 > 0$, it follows that $f(u) = 0$ has one real positive root and one real negative root. Let $u = \lambda$ be the positive root then, if the angle between the asymptotes is 2α, we have

$$\alpha = \int_0^\lambda \left\{\frac{1}{p^2} - u^2 + \frac{2\kappa m}{c^2} u^3 - \frac{\kappa\epsilon^2}{c^4} u^4\right\}^{-\frac{1}{2}} du.$$

Writing $u = \lambda v$, and remembering that $f(\lambda) = 0$, this gives

$$\alpha = \int_0^1 (1-v)^{-\frac{1}{2}} \left\{1 + v - \frac{2\kappa m}{c^2} \lambda (1 + v + v^2) + \frac{\kappa\epsilon^2}{c^4} \lambda^2 (1+v)(1+v^2)\right\}^{-\frac{1}{2}} dv. \quad (40)$$

For rays which approach the electron very closely, p is very small and λ is very large, and, for sufficiently large values of λ (40), approximates to

$$\alpha \doteq \frac{1}{\lambda} \frac{c^2}{\kappa^{\frac{1}{2}}\epsilon} \int_0^1 \frac{dv}{\sqrt{(1-v^4)}},$$

which tends to zero as λ increases. Hence rays which, if undisturbed, would pass very close indeed to the electron, are reflected back practically along their original direction.

For larger values of p, we may conveniently refer back to equation (39). Differentiating with respect to ϕ, we obtain

$$\frac{d^2u}{d\phi^2} + u = \frac{3\kappa m}{c^2} u^2 - \frac{2\kappa\epsilon^2}{c^4} u^3. \quad (41)$$

Allowing for the difference of the units employed, this agrees with the usual equation for the deflection of a ray of light, except for the presence of the last term, which gives the influence of the electric field.

The equation (41) may be integrated by successive approximation in the usual way if u is so small that the terms on the right-hand side are small compared with u. In this way we obtain as the total deflection of the ray

$$\frac{4\kappa m}{c^2 p} - \frac{3\pi\kappa\epsilon^2}{4c^4p^2}. \quad (42)$$

The first term is the well-known Einstein deflection, and the second term shows that the deflection is reduced by the electric field. This approximation will hold for sufficiently large values of p and we have already examined the case of very small values of p. The intermediate cases may be studied by means of (39) if need should arise.

§ 5. *The Influence of the Sun's Electric Field on the Crucial Phenomena of Einstein's Theory.*

In accordance with these results the deflection of a ray of light by the Sun and the predicted displacement of the lines of the Solar spectrum will be influenced by the Sun's electric field.

The ratio of the wave-length of the radiation from a terrestrial atom to that of the radiation from a similar atom in the Sun is, in accordance with the usually accepted theory, given by $\sqrt{\gamma}$. Now it appears from (35) that the effect of any electric field which the Sun may have, whatever its sign, will tend to counteract the effect of the Sun's mass, and we may inquire what strength of field would be necessary to produce exact compensation. For this purpose we must have $\gamma = 1$ at the Sun's surface, or from (35),

$$\epsilon^2 = 2mc^2r.$$

Such a charge on the Sun, if distributed symmetrically, would give at the Sun's surface a field of intensity

$$c\sqrt{\frac{2m}{r^3}} \text{ c.g.s. units} \quad \text{or} \quad c^2\sqrt{\frac{2m}{r^2}}\, 10^{-8} \text{ volts/cm.}$$

The quantity under the square root being of the same order as the Sun's mean density, this requires a potential gradient of the order of 10^{13} volts. per cm., which is far too high to be admissible. The solution to the vexed question of the displacement of the spectrum lines is not to be found in this direction.

In the same way it may be shown that, for any possible value of the Sun's electric field, the second term in (42) is very small compared with the first, and the electric field will not exert any measurable effect on the bending of the rays of light in the gravitational field of the Sun.